Denotational semantics
Exercises

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Supervision 1

1 Introduction

1. Solve exercise 1.3.1. Don't spend too much time on this exercise. A straightforward, verbose, solution shouldn't take more than 60 lines of SML.
2. Solve exercise 1.3.2.
3. Solve exercise 1.3.3.
4. Solve exercise 1.3.4.

2 Domains

1. Solve exercise 2.5.1.
2. Solve exercise 2.5.2. As the claims in slide 27 are already proved in the notes, there is no need to prove them.
3. We say that a chain, $x_0 \sqsubseteq x_1 \sqsubseteq x_2 \sqsubseteq \ldots$, is *eventually constant* if there exists a natural number k such that for all natural numbers $n \geq k$, we have $x_n = x_k$.
   (a) Show that every eventually constant chain has a lub.
   (b) Deduce that every finite poset is a cpo.
   (c) Show that every monotone function preserves lubs of eventually constant chains.
   (d) Deduce the following result: Let D, E be cpos such that all chains in D are eventually constant. All monotone functions $f : D \to E$ are continuous.

4. Solve exercise 2.5.3.

5. Let D, D' be domains. We say that a function $f : D \to D'$ is a *continuous isomorphism* if it is continuous, bijective, and its inverse $f^{-1} : D' \to D$ is also continuous.

   (a) Show that if f is continuous and bijective, and f^{-1} is monotone, then f is a continuous isomorphism.

   (b) Find an example for a continuous and bijective f that is not a continuous isomorphism.

6. (Due to Meseguer) Let $A \coloneqq \{a_0, a_1, \ldots\}$, $B \coloneqq \{b_0, b_1, \ldots\}$ and $\{\infty\}$ be pairwise disjoint sets. Define a binary relation $\sqsubseteq$ over $D_0 \coloneqq A \cup B \cup \{\infty\}$ by $v \sqsubseteq w$ if and only if:

   - $w = \infty$, or
   - $w = v = b_n$, or
   - $w = b_n$ and $v = a_m$, for some $n \geq m \geq 0$, or
   - $w = a_n$ and $v = a_m$, for some $n \geq m \geq 0$.

   (a) Show that $\langle D_0, \sqsubseteq \rangle$ is a domain.

   (b) (optional) Draw a Hasse diagram for D_0.

   (c) Let $\langle D, \sqsubseteq \rangle$ be any domain, and $f, g : D_0 \to D$ two continuous functions. Show that if, for all n, $f(b_n) = g(b_n)$, then also $f(\infty) = g(\infty)$.

Supervision 2

3 Constructions on domains

1. (a) Let X be a set. Show that the discrete cpo $(X, =)$ is a cpo, and that the flat domain $X_\perp$ is a domain. (cf. Slide 31.)

   (b) Let X be a set and D a domain. Show that every monotone function $f : X_\perp \to D$ is continuous.

   (c) Let $f : X \rightharpoonup Y$ be a partial function between two sets X, Y. Show that $f_\perp : X_\perp \to Y_\perp$ is continuous and strict. (cf. Proposition 3.1.1.)

2. (a) Show that the product and dependent product of domains is a domain. (cf. Slide 32 and Definition 3.2.3.)

   (b) Show that the function $if : \mathbb{B}_\perp \times (D \times D) \to D$ is continuous. (cf. Proposition 3.2.2.)

3. (a) Let D, E be cpos. Show that the function cpo $(D \to E, \sqsubseteq)$ is a cpo, and justify the following rule (cf. Slide 35.):

$$\frac{f \sqsubseteq_{(D \to E)} g \qquad x \sqsubseteq_D y}{f(x) \sqsubseteq_E g(y)}$$

(b) Let D, E be cpos. Find a necessary and sufficient condition on D and E such that $D \to E$ is a domain.

(c) Let D, E, F be domains. Recall the composition operation:

$$\circ : (E \to F) \times (D \to E) \to (D \to F)$$
$$g \circ f(d) \coloneqq g(f(d))$$

What do you need to do to show that $\circ$ is well-defined? Show it.

(d) Show that $\circ$ is continuous in each argument.

(e) Deduce the proposition on Slide 37.

4. Solve exercise 3.4.2.

5. (a) Solve exercise 3.4.3. What is the bijection?

   (b) Prove or disprove: there is a continuous isomorphism from $\Omega \to \{\top\}_\perp$ to Ω.

4 Scott induction

1. Show that the following subsets are chain-closed:

   (a) $\{\langle x, y\rangle \in D \times D | x \sqsubseteq y\}$, for every cpo D.

   (b) $\downarrow(d) \coloneqq \{x \in D | x \sqsubseteq d\}$ for every d in any cpo D.

   (c) $f^{-1}[S] \coloneqq \{x \in D | f(x) \in S\}$, for every continuous function $f : D \to E$ and chain-closed subset S of E.

   (d) $S \cup T$ for every chain-closed subsets S, T of any cpo D.

   (e) $\bigcap_{i \in I} S_i$ for every I-indexed family of chain-closed subsets S_i of any cpo D.

   (f) $\{\langle x, y\rangle \in D \times D | x = y\}$, for every cpo D.

2. Solve exercise 4.4.2: For every subset $S \subseteq D \times D'$, and for every $d \in D$ and $d' \in D'$, define:

$$S_d \coloneqq \{d' \in D' | \langle d, d'\rangle \in S\}$$
$$S^{d'} \coloneqq \{d \in D | \langle d, d'\rangle \in S\}$$

   Give an example of a subset S that is not chain-closed, yet for every $d \in D$ and $d' \in D'$, both S_d and $S^{d'}$ are chain-closed. (Compare this with the property of continuous functions given on Slide 33.)

3. Let D, D' be domains. Show that a monotone function $f : D \to D'$ is continuous and strict if and only if, for every admissible subset $S' \subseteq D'$, the inverse image $f^{-1}[S'] \subseteq D$ is admissible.

4. Let $\langle D, \sqsubseteq\rangle$ be a domain and $X \subseteq D$ a subset. We define the *admissible closure* of X in D as the smallest admissible subset containing X:

$$\operatorname{Cl} X := \bigcap_{\substack{X \subseteq S \subseteq D \\ \text{is admissible}}} S$$

   (a) Show that $\operatorname{Cl} X$ is an admissible subset.

   (b) Show that $\operatorname{Cl} X$ with the order induced by $\sqsubseteq$ is a domain, and the inclusion map $\iota : \operatorname{Cl} X \to D$, $\iota(x) := x$ is continuous and strict.

   (c) Let D' be any other domain, and $f, g : D \to D'$ be two strict continuous functions that agree on X, i.e.: for all $x \in X$, $f(x) = g(x)$. Show that f and g agree on $\operatorname{Cl} X$.

   (d) Let D be a domain and $X \subseteq D$ a subset of it. Define:

$$Y := X \cup \{\bot\} \cup \left\{ \bigsqcup_n x_n \,\middle|\, x_n \text{ is a chain in } X \right\}$$

   Show that $Y \subseteq \operatorname{Cl} X$, and find an example in which $Y \neq \operatorname{Cl} X$.

5. Let $\langle D_1, \sqsubseteq_1\rangle$, $\langle D_2, \sqsubseteq_2\rangle$ be two domains, and $f : D_1 \to D_2$ a strict continuous function between them. We say that:

   - f is *full* when, for all d, d' in D_1, $f(d) \sqsubseteq f(d')$ entails $d \sqsubseteq d'$.
   - f is *dense* when the image $f[D_1] := \{f(d) | d \in D_1\}$ satisfies:

$$\operatorname{Cl} f[D_1] = D_2$$

   (a) Show that every full function is injective.

   (b) Find a dense function that is *not* surjective.

   (c) Let $f : D \to D'$ be a strict continuous function. Find a domain D_f, a dense function $e : D \to D_f$ and a full function $m : D_f \to D'$, such that for $f = m \circ e$, i.e., for all $d \in D$, $f(d) = m(e(d))$.

6. (optional) Let D, D', D_1, and D_2 be domains, $e_1 : D \to D_1$, $e_2 : D \to D_2$ dense functions, and $m_1 : D_1 \to D'$ $m_2 : D_2 \to D'$ full functions, such that:

$$m_1 \circ e_1 = m_2 \circ e_2$$

   Define:

$$S_1 := \{d_1 \in D_1 | \exists d_2 \in D_2 : m_1(d_1) = m_2(d_2)\}$$

   (a) Show that S_1 is admissible:

   (b) Show that $S_1 = D_1$.

   (c) Define a continuous strict function $h : D_1 \to D_2$, such that $h \circ e_1 = e_2$ and $m_1 = m_2 \circ h$

   (d) Show that h is a continuous isomorphism.

7. (optional, Adapted from Plotkin's Pisa notes) Let $f, g : D \to D$ be two continuous functions over a domain D.

   (a) Show that $f(\mathrm{fix}(g \circ f)) = \mathrm{fix}(f \circ g)$.
   Hint: consider the admissible subset $\{x \in D | f(x) \sqsubseteq \mathrm{fix}(f \circ g)\}$.

   (b) Show that if $f(\perp) = g(\perp)$ and $f \circ g = g \circ f$, then:

   $$\mathrm{fix}\, f = \mathrm{fix}\, g = \mathrm{fix}(f \circ g)$$

   Deduce that for all continuous functions $f : D \to D$, $\mathrm{fix}\, f^2 = \mathrm{fix}\, f$.

   (c) Dear students: We need your help! Plotkin no longer remembers the proof for the following clause, and I (Ohad) couldn't easily reproduce it. If you can do it yourself correctly, please get in touch!
   Show that if $f(\perp) = g(\perp)$ and $f \circ f \circ g = g \circ f$, then $\mathrm{fix}\, f = \mathrm{fix}\, g$.

Supervision 3

5 PCF: syntax and semantics

1. Solve exercise 5.7.1.
2. Solve exercise 5.7.2.
3. Solve exercise 5.7.3.
4. Solve exercise 6.5.1
5. Solve exercise 6.5.2.
6. Solve clause (c) **only** of Question 15, Paper 9, 2005.
   Note: please prove your answer.
7. Solve Question 6, from Paper 9, 2009.
   Note: no need to solve clause (a)(i).

6 Logical relations and full abstraction

1. Solve exercises 7.4.1.
2. Solve exercises 7.4.2.
3. Solve exercises 7.4.3.
4. Solve exercises 8.4.1.
5. Solve exercises 8.4.2.
6. Solve exercises 8.4.3.