A modern perspective on the O'Hearn-Riecke model

Extended Abstract

Ohad Kammar
University of Edinburgh
Scotland

Shin-ya Katsumata
National Institute of Informatics
Tokyo, Japan

1. Introduction

Around the same time that Hyland and Ong [5, 6] and Abramsky, Jagadeesan, and Malacaria [1, 2] gave a fully abstract game semantics to PCF, O'Hearn and Riecke [14] also gave a fully abstract model combining domain-theoretic and relational techniques. The games models give an intensional description of the semantics through a careful analysis of the interaction of a program with its environment. Once the semantics characterises the appropriate intensional interaction, one quotients the model through the extensional collapse process to get a fully abstract model. The O'Hearn-Riecke (OHR) model starts out with the usual extensional, domain-theoretic model, and then uses logical relations to cut out junk from the model. Game semantics has since been extended to deal with a wide spectrum of effects, whereas the O'Hearn-Riecke model remained relatively untouched, notably excepting Stark [16].

In the proposed talk, we will describe our ongoing work analysing the OHR model. We hope that, 25 years later, we can extend it to account for other effects. This work is at an early stage. We hope to use the workshop to stimulate informal discussion about directions for further questions, as well as learn folklore about results concerning the OHR model.

We structure our development from the modern perspective on a programming language with computational effects [10, 13]: a category for values (pre-domains), a strong monad over it, with call-by-name semantics taking place in (a suitable subcategory of) the Eilenberg-Moore category for this monad.

2. Values/pre-domains

Before presenting the value part of the OHR model, we consider a simpler construction on the category $\mathbf{Set}$ of sets and functions.

Example 1 (Binary endo-relations)**.** The category $\mathbf{ERel}$ has as objects pairs $R = \langle \underline{R}, \dot{R} \rangle$ consisting of a set $\underline{R}$ and a binary endo-relation $\dot{R} \subseteq \underline{R}^2$ (relation, for brevity). A morphism $f : R \to S$ is a function $f : \underline{R} \to \underline{S}$ preserving the relation:

$$\langle x_1, x_2 \rangle \in \dot{R} \quad \Longrightarrow \quad \langle f\, x_1, f\, x_2 \rangle \in \dot{S}$$

The category $\mathbf{ERel}$ is the change-of-base of the subobject fibration along the functor multiplying each set/function with itself. This category is cartesian closed, whose exponential is given as in Fig. 1.

Let R be a binary relation. We say that an element $x \in \underline{R}$ is *R-invariant* [15] when $\langle x, x \rangle \in \dot{R}$. We say that R is *concrete* when every element in $\underline{R}$ is invariant, i.e., when R is reflexive. Let $\mathbf{RRel} \hookrightarrow \mathbf{ERel}$ be the full subcategory consisting of the concrete/reflexive relations. This embedding has both adjoints. The left adjoint $C : \mathbf{ERel} \to \mathbf{RRel}$ simply adds the diagonal:

$$CR := \left\langle \underline{R}, \dot{R} \cup \{\langle x, x \rangle | x \in \underline{R}\} \right\rangle$$

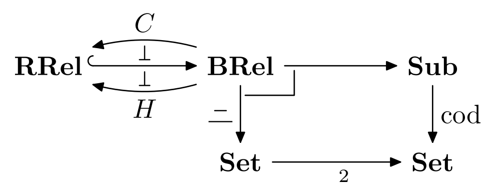
$$\mathbf{ERel} : \underline{S^R} = \mathbf{Set}(\underline{R}, \underline{S}) \qquad \mathbf{RRel} : \underline{S^R} = \mathbf{RRel}(R, S)$$
$$\text{Both} \;: \dot{S^R} = \left\{ \langle f_1, f_2 \rangle \middle| \langle x_1, x_2 \rangle \in \dot{R} \Rightarrow \langle f_1\, x_1, f_2\, x_2 \rangle \in \dot{S} \right\}$$

Figure 1. Exponentials in $\mathbf{ERel}$ and $\mathbf{RRel}$

The right adjoint $H : \mathbf{ERel} \to \mathbf{RRel}$ restricts the relation to its invariant elements, i.e., its *reflexive centre*:

$$HR := \left\langle \left\{ x \in \underline{R} \middle| \langle x, x \rangle \in \dot{R} \right\}, \dot{R} \cap \underline{(HR)}^2 \right\rangle$$

We summarise this situation in a diagram (in $\mathbf{CAT}$):



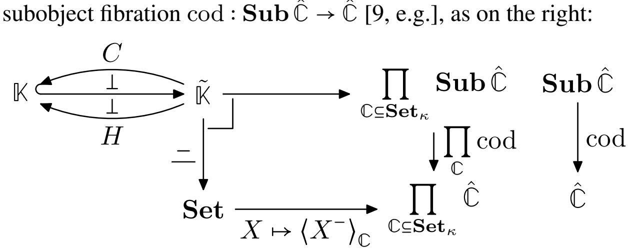
Using the coreflection $J \dashv H$, the following becomes a cartesian closed structure on $\mathbf{RRel}$ [3, Proposition 27.9]:

$$R \times S := H(JR \times JS) \quad S^R := H\left((JS)^{JR}\right).$$

Fig. 1 compares exponentials in $\mathbf{ERel}$ and $\mathbf{RRel}$, where in general $\mathbf{RRel}(R, S) \subsetneq \mathbf{Set}(\underline{R}, \underline{S})$ □

The OHR model generalise this situation in two ways. The first one is to move to Kripke relations of varying arity, and the second is to move to ω-chain-closed relations.

Example 2 (Kripke relations of varying arity)**.** Fix a cardinal κ bounding the arity of the relations. For finitary relations, we use the countable cardinal $\kappa := \aleph_0$. Let $\mathbf{Set}_\kappa$ be the (small) full subcategory of $\mathbf{Set}$ consisting of the hereditarily κ-small sets. For each subcategory $\mathbb{C} \subseteq \mathbf{Set}_\kappa$, consider the presheaf category $\hat{\mathbb{C}} := [\mathbb{C}^{\mathrm{op}}, \mathbf{Set}]$. From the general theory of fibrations, $\mathbf{Sub}\,\hat{\mathbb{C}}$ has a bi-cartesian closed structure that is strictly preserved by the subobject fibration $\mathrm{cod} : \mathbf{Sub}\,\hat{\mathbb{C}} \to \hat{\mathbb{C}}$ [9, e.g.], as on the right:



Taking the (small) product in $\mathbf{CAT}$, ranging over $\mathbb{C}$, we have a bi-fibration with the same properties (to the left of cod), for which we can take the change-of-base along the functor sending each X to the diagonal presheaf $i \in \mathbb{C} \mapsto X^i$ (further to the left). Concretely, the total category $\tilde{\mathbb{K}}$ of *abstract Kripke logical relations of varying arity* has as objects $R = \langle \underline{R}, \dot{R}_{-} \rangle$ consisting of a set $\underline{R}$ together with, for every small subcategory $\mathbb{C} \subseteq \mathbf{Set}_\kappa$, and every object $w \in \mathbb{C}$, a relation $\dot{R}_{\mathbb{C}} w \subseteq \underline{R}^w$, such that for all $\rho : w \leftarrow u$ in $\mathbb{C}$, the function $\underline{X}^\rho$ respects the relations, i.e.:

$$\langle r_i \rangle_{i \in w} \in \dot{R}_{\mathbb{C}} w \quad \Longrightarrow \quad \langle r_{\rho\, j} \rangle_{j \in u} \in \dot{R}_{\mathbb{C}} u$$

Morphisms $f : R \to S$ are functions $f : \underline{R} \to \underline{S}$ s.t.:

$$\forall \mathbb{C}, w \in \mathbb{C} : \quad \langle r_i \rangle_{i \in w} \in \dot{R}_{\mathbb{C}} w \quad \Longrightarrow \quad \langle f\, r_i \rangle_{i \in w} \in \dot{S}_{\mathbb{C}} w$$

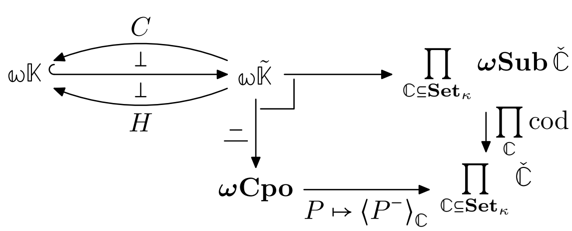
This situation generalises binary relations: choosing $\mathbb{P}$ to be the one-object subcategory consisting of $\{0, 1\}$ with only the identity function on it, we get a forgetful functor $U : \tilde{\mathbb{K}} \to \mathbf{ERel}$ given by $R \mapsto \langle \underline{R}, R_{\mathbb{P}} \{0, 1\} \rangle$. Finally, we say that $R \in \tilde{\mathbb{K}}$ is *concrete* when every $r \in \underline{R}$ is R-invariant: for all $\mathbb{C}$, and $w \in \mathbb{C}$, we have $\Delta_w r := \langle r \rangle_{i \in w} \in \dot{R}_{\mathbb{C}} w$. We take $\mathbb{K}$ to be the full subcategory of the concrete relations. The inclusion $J : \mathbb{K} \hookrightarrow \tilde{\mathbb{K}}$ has a left adjoint C, adding the diagonal to each relation, as well as a right adjoint H:

$$\underline{HR} := \left\{ r \in \underline{R} \middle| \forall \mathbb{C}, w. \Delta_w r \in \dot{R}_{\mathbb{C}} w \right\} \qquad (\dot{HR})_{\mathbb{C}} w := \dot{R}_{\mathbb{C}} w \cap \underline{HR}^w$$

From the general theory of fibrations [9], $\tilde{\mathbb{K}}$ is bi-cartesian closed, and as in Ex. 1, $\mathbb{K}$ is bi-cartesian closed. This bi-cartesian closed structure is *not* preserved by $\underline{J-}$. ☐

Recall that a *pre-domain* is a poset $P = \langle \underline{P}, \leq \rangle$ where every increasing sequence $\langle p_n \rangle_{n \in \omega}$ indexed by the ordinal ω (an ω-chain) has a least upper bound (lub), denoted $\bigvee_n p_n$. Let $\boldsymbol{\omega}\mathbf{Cpo}$ be the category of pre-domains and *Scott continuous* functions: monotone functions preserving lubs of ω-chains. This category is bi-cartesian closed and interprets values as usual in domain theory.

Example 3 (ω-chain-closed relations)**.** Let $\boldsymbol{\omega}\mathbf{Sub}$ be the full-subcategory of $\mathbf{Sub}\,\boldsymbol{\omega}\mathbf{Cpo}$ consisting of the ω-chain-closed subsets with order-reflecting inclusions as objects, and consider the codomain bi-fibration $\mathrm{cod} : \boldsymbol{\omega}\mathbf{Sub} \to \boldsymbol{\omega}\mathbf{Cpo}$. Generalising to Kripke structures, given a small subcategory $\mathbb{C} \subseteq \mathbf{Set}_\kappa$, consider the $\boldsymbol{\omega}\mathbf{Cpo}$-presheaf category $\check{\mathbb{C}} := [\mathbb{C}, \boldsymbol{\omega}\mathbf{Cpo}]$. We take $\boldsymbol{\omega}\mathbf{Sub}\,\check{\mathbb{C}}$ as a skeleton of the full subcategory of $\mathbf{Sub}\,\check{\mathbb{C}}$ with the component-wise order-reflecting subobjects. From the theory of topological functor [3, Ch. 22–23], it is fibre-wise bi-cartesian closed, and the fibration preserves the (total) bi-cartesian closed structure. We reproduce the situation from Ex. 2:



where $\omega\mathbb{K}$ is given by imposing the concreteness condition. Again, the coreflecttion makes $\omega\mathbb{K}$ bi-cartesian closed. ☐

3. Computations/monads

Let $L = \langle \underline{L} : \boldsymbol{\omega}\mathbf{Cpo} \to \boldsymbol{\omega}\mathbf{Cpo}, \mathrm{return}, \gg\!= \rangle$ be the lifting monad, adjoining to each ω-cpo a new least element $\bot$. Thanks to the adjunction $J \dashv H$, we can transform monads M over $\omega\tilde{\mathbb{K}}$ to monads T over $\omega\mathbb{K}$ by setting $\underline{T} := H\underline{M}J$. In particular, we will transform *monadic liftings* of L.

Example 4 (Hermida)**.** Every monad can be lifted along a fibration for logical relations by taking the direct image of the unit [4]. In our case, as the unit is injective, transforming this lifting along the adjunction yields the identity monad on $\omega\mathbb{K}$. ☐

Example 5 (free lifting)**.** Taking the smallest lifting that is both compatible with the image and contains the least element $\bot$, i.e., making each lifting $\dot{M}$ an *admissible* subset of $\underline{L}X$. This is a special case of the *free lifting* [7]. We use this lifting. ☐

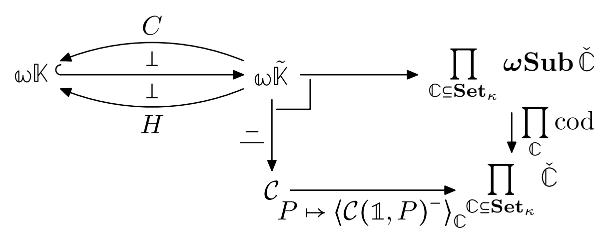
The Eilenberg-Moore category for T consists of the admissible Kripke relations of varying arity, which is the crux of the OHR construction. To complete it, we note that in order to interpret a call-by-*name* language with the natural numbers as base type, we need an appropriate T-algebra. OHR bakes this choice of algebra into their category, but we want to separate it into the model structure.

4. Definability

Let τ range over PCF types. In the final step in the construction, we use Katsumata's [8] definability characterisation using $\top\top$-lifting. The $\top\top$-lifting of L to $\tilde{\mathbb{K}}$ characterises the elements (approximated by) definable elements. As the free lifting is contained in any lifting containing $\bot$, and contains all the definable elements, we deduce that every element in $T[\![\tau]\!]$ can be approximated by definable elements, giving us the usual full-abstraction result. To use the $\top\top$-lifting, we need to choose the cardinal κ to be large enough so that each $[\![\tau]\!]$ is κ-small. However, for PCF, OHR replace the definable elements by Milner's finite definable approximations method [11].

5. Prospects

We would like to transport this account to languages with arbitrary effects. As a starting point, we will consider a model $\mathcal{C}$ for the programming language at hand, requiring it to be $\boldsymbol{\omega}\mathbf{Cpo}$-enriched. We will then use the sub-scone [12] to reconstruct a generalisation:



further assuming $\mathcal{C}$ is sufficiently complete well-behaved for the adjoints to exist. As all of the constructions we have used, including the free lifting, and definability via $\top\top$-lifting, are valid in this situation, we hope to break new ground in this general setting.

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