10 Type structure

The type combinators — tuples, variants, functions, and their recursive combinations — are the basic building-blocks of compositional programming, and we can similarly use them as building-blocks for statistical modelling. This sheet covers classical material in the semantics of type structure, specialised for quasi-Borel spaces.

We've already seen the qbs constructions that do much of the low-level work: the *ground* combinators — products and coproducts.

Given a *sequence* of spaces spaces $A_1, \ldots, A_n$, their:

- *tuple* space of is the product $A_1 \times \cdots \times A_n$ with elements the n-tuples $\langle a_1, \ldots, a_n \rangle$ (Ex.7.4).
- *variant* space is the coproduct $A_1 \amalg \cdots \amalg A_n$ with elements $\iota_i a$, $1 \leq i \leq n$, $a \in \llcorner A_i \lrcorner$ (Ex.7.5).

When modelling, as with programming, using positional tuples and variants can be tedious and confusing, and doesn't scale well to large or structured collections of spaces. So we also introduce the indexed versions of tuples, called *records*, and indexed variants. Given an I-indexed *set* of spaces $\langle A_i \rangle_{i \in I}$, their:

- *record* space is the I-indexed product $\prod_{i \in I} A_i$.
- *variant* space is the I-indexed coproduct $\coprod_{i \in I} A_i$.

The structure of these spaces depends on a set rather than a sequence, and so only the labels matter, not their order. We'll therefore use set-comprehension-like notation for the indexed versions, to emphasise that the order of components doesn't matter, only their indices. So when $I = \{\ell_1, \ldots, \ell_n\}$, we'll use:

- $\langle \ell_1 : A_{\ell_1}, \ldots, \ell_n : A_{\ell_n} \rangle \coloneqq \prod_{\ell \in I} A_\ell$
  with element records $\langle \ell_1 : a_{\ell_1}, \ldots, \ell_n : a_{\ell_n} \rangle \coloneqq \langle a_\ell \rangle_{\ell \in I}$.
- $\{\ell_1 : A_{\ell_1} \mid \ldots \mid \ell_n : A_{\ell_n}\} \coloneqq \coprod_{\ell \in I} A_\ell$ with 'constructor-headed' elements $\ell\, a \coloneqq \iota_\ell a$ for $\ell \in I$.

References