8 Borel subspaces

The central notion in measure theory is that of a measurable subset — it is the defining concept of a measurable space. With quasi-Borel spaces, measurable subsets are a derived notion, but take a nonetheless central role.

◸**8.1.** A *measurable*, or *Borel*, subset in a qbs A is a subset $U \subseteq \llcorner A \lrcorner$ such that the preimage under every random element $\alpha \in \mathcal{R}_A$ is a Borel subset of the reals: $\alpha^{-1}[U] \in \mathcal{B}$. We denote by $\mathcal{B}_A$ the set of Borel subsets of A.

- Show that the measurable sets $\mathcal{B}_A$ in a qbs A form a σ-algebra, and every random element is measurable w.r.t. this σ-algebra.

  We denote the resulting measurable space by $\overset{\mathbf{Meas}}{\ulcorner A \urcorner} := \left\langle \underset{\mathbf{Set}}{\llcorner A \lrcorner}, \mathcal{B}_A \right\rangle$, and call it the *free* measurable space over A.
- Show that $U \subseteq \llcorner A \lrcorner$ is measurable iff its indicator function $[- \in U] : A \to \overset{\mathbf{Qbs}}{\ulcorner \mathbb{2} \urcorner}$ is a qbs morphism from A into the discrete qbs on the two-element set. ◿

◸**8.2.** Find the Borel sets of the discrete qbs $\overset{\mathbf{Qbs}}{\ulcorner \mathbb{2} \urcorner}$ and the indiscrete qbs $\underset{\mathbf{Qbs}}{\llcorner \mathbb{2} \lrcorner}$ on two elements. Generalise this result to the discrete and indiscrete qbses over any set X. ◿

◸**8.3.** Show that the Borel subsets of $\mathbb{R}$ in the standard sense coincide with the measurable subsets of the qbs $\mathbb{R}$. ◿

◸**8.4.** Let A be a qbs and $X \subseteq \llcorner A \lrcorner$ be a subset.

- Show that if $U \subseteq \llcorner A \lrcorner$ is Borel in A, then $U \cap X$ is Borel in the subspace X:

$$U \in \mathcal{B}_A \implies U \cap X \in \mathcal{B}_X$$

- Show that if X is itself a Borel subset, then $\mathcal{B}_X \subseteq \mathcal{B}_A$.
- Show that the previous clause may fail if X is not Borel. ◿

The Borel subsets of a subspace can be quite different from the Borel subsets of its superspace. For example, we may have a Borel subset $V \in \mathcal{B}_X$ of the subspace that is not of the form $U \cap X$ for any Borel subset $U \in \mathcal{B}_A$ of the superspace.

Here's the intuition:

- A subset U in a qbs is measurable unless there is some random element that stops it from being measurable by mapping U onto a non-Borel inverse image.
- 'Wild' random elements may not factor through a subspace embedding $X \hookrightarrow A$.
- So a subspace may have more Borel subsets in X than in its superspace.

If you want to see this intuition playing out, here is how to construct a counter-example:

◸**8.5.** Let $C_1 \subseteq \mathbb{R}$ be a non-Borel subset and $C_2 := \mathbb{R} \setminus C_1$ its complement, also non-Borel. Let $\mathbb{3} := \{0, 1, 2\}$ be a three-element set, and define two primitive random elements $\alpha_i : \mathbb{R} \to \mathbb{3}$:

$$\alpha_0 r := \begin{cases} r \in C_1 : 0 \\ r \in C_2 : 2 \end{cases} \qquad \alpha_1 r := \begin{cases} r \in C_1 : 1 \\ r \in C_2 : 2 \end{cases}$$

Take $A := \langle \mathbb{3}, \mathrm{Cl}_{\mathrm{qbs}} \{\alpha_0, \alpha_1\} \rangle$ to be the qbs over $\mathbb{3}$ with the smallest metaphorology (see Ex.7.9) containing α_0 and α_1, and take $X := \mathbb{2} \subseteq \mathbb{3}$.

- Show that $X, \{0\}, \{0,2\} \notin \mathcal{B}_A$ are not Borel subsets in A.
- Show that if $\alpha \in \mathcal{R}_A$ is a random element in A, then either α is σ-simple or $2 \in \mathrm{Im}(\alpha)$.
- Show that $\{0\} \in \mathcal{B}_X$ is a Borel subset of the subspace X. ⊿

◸**8.6.** Let $f : A \to B$ be a qbs morphism. Show that:

- The inverse image under f restricts to a function $\mathcal{B}_f : \mathcal{B}_B \to \mathcal{B}_A$.
- The underlying function $\llcorner f \lrcorner_{\mathbf{Set}}$ is a measurable function $\ulcorner f \urcorner^{\mathbf{Meas}} : \ulcorner A \urcorner^{\mathbf{Meas}} \to \ulcorner B \urcorner^{\mathbf{Meas}}$. ⊿

The collection of Borel sets has a universal property: it allows us to connect measurable spaces with quasi-Borel spaces as follows:

◸**8.7.** For a measurable space M, define its set of *random elements* by $\mathcal{R}_M \coloneqq \mathbf{Meas}(\mathbb{R}, M)$.

- Show that $\mathcal{R}_M$ is a metaphorology, that is, $\llcorner M \lrcorner_{\mathbf{Qbs}} \coloneqq \left\langle \llcorner M \lrcorner_{\mathbf{Set}}, \mathcal{R}_M \right\rangle$ is a qbs.
- For every measurable function $f : M \to N$ between measurable spaces, show that its underlying function is a qbs morphism $\llcorner f \lrcorner_{\mathbf{Qbs}} : \llcorner M \lrcorner_{\mathbf{Qbs}} \to \llcorner N \lrcorner_{\mathbf{Qbs}}$.
- Noticing that $\llcorner - \lrcorner_{\mathbf{Qbs}} : \mathbf{Meas} \to \mathbf{Qbs}$ is a (faithful) functor, show that it has a left adjoint equipping a qbs with its set of Borel subsets: $\ulcorner - \urcorner^{\mathbf{Meas}} \dashv \llcorner - \lrcorner_{\mathbf{Qbs}}$. ⊿

◸**8.8.** The free qbs functor $\ulcorner - \urcorner^{\mathbf{Qbs}} : \mathbf{Set} \to \mathbf{Qbs}$ doesn't preserve countable products. ⊿

This point is a natural place to stop, but if you're having fun with this material, then the rest of this sheet studies the relationships between natural notions of 'subspace'.

- $m : A \rightarrowtail B$ Monomorphisms: injective qbs morphisms.
- $m : A \hookrightarrow B$ Subspace embedding: injective on elements and surjective on random-elements that factor through the image.
- $m : A \mathrel{\hookrightarrow\!\!\!\!\rightarrowtail} B$ Borel injections: monomorphisms whose image is a Borel subset.
- $m : A \mathrel{\lhook\joinrel\hookrightarrow} B$ Borel embeddings: subspace embeddings whose image is a Borel subset.

We establish their following mutual relationships, where all inclusions are proper:

$$
\begin{tikzcd}
& \textbf{mono} & \\
& & \textbf{subspace embedding} \arrow[ul, hook] \arrow[d, equal] \\
\textbf{Borel mono} \arrow[uur, hook] & & \textbf{strong mono} \arrow[d, equal] \\
& & \textbf{regular mono} \\
& \textbf{Borel embedding} \arrow[uul, hook] \arrow[ur, hook] &
\end{tikzcd}
$$

◸**8.9.** Place the following injections in the hierarchy of monomorphisms above:

- The injection $\top \coloneqq \lambda \star . 1 : \ulcorner \mathbb{1} \urcorner^{\mathbf{Qbs}} \rightarrowtail \ulcorner \mathbb{2} \urcorner^{\mathbf{Qbs}}$.
- The injection $\lambda x.x : \ulcorner \mathbb{2} \urcorner^{\mathbf{Qbs}} \rightarrowtail \llcorner \mathbb{2} \lrcorner_{\mathbf{Qbs}}$.

- The injection $\lambda x.x : \ulcorner \overset{\mathbf{Qbs}}{2} \urcorner \rightarrowtail \llcorner \underset{\mathbf{Qbs}}{3} \lrcorner$.
- The (subspace) inclusion $\lambda x.x : C \hookrightarrow \mathbb{R}$ where C is a non-Borel subset of $\mathbb{R}$. ⊿

◸**8.10.** Let $m : S \to A$ be a qbs morphism. Show that the following are equivalent:

- m is a subspace embedding, i.e.: there is a subset $X \subseteq \llcorner A \lrcorner$ and an isomorphism $m' : B \xrightarrow{\cong} X$ satisfying:

$$\begin{tikzcd} S \arrow[dr, hook, "m"] \arrow[d, "m'"', "\cong"] & \\ X \arrow[r, phantom, "="] \arrow[r, hook, "\lambda x.x"'] & A \end{tikzcd}$$

- m is *right-orthogonal* to every empimorphism $e : B \twoheadrightarrow C$: for every commuting square as on the left, there is a unique morphism $h : C \to S$ commuting the triangles on the right:

$$\begin{tikzcd} B \arrow[r, two heads, "e"] \arrow[d, "f"'] \arrow[dr, phantom, "="] & C \arrow[d, "g"] \\ S \arrow[r, hook, "m"'] & A \end{tikzcd} \quad\Longrightarrow\quad \begin{tikzcd} B \arrow[r, two heads, "e"] \arrow[d, "f"'] & C \arrow[d, "g"] \arrow[dl, dashed, "h"'] \\ S \arrow[r, hook, "m"'] & A \end{tikzcd}$$

(Morphisms that have this property are called *strong monomorphisms*.)

- m is an *equaliser* of some parallel pair of morphisms $f, g : A \to B$:
  - m *equalises* f and g:

$$\begin{tikzcd} & A \arrow[dr, "f"] & \\ S \arrow[ur, "m"] \arrow[dr, "m"'] \arrow[rr, phantom, "="] & & B \\ & A \arrow[ur, "g"'] & \end{tikzcd}$$

  - and every equalising morphism $e : C \to A$ factors uniquely through m:

$$\begin{tikzcd} & A \arrow[dr, "f"] & \\ E \arrow[ur, bend left, "e"] \arrow[dr, bend right, "e"'] \arrow[rr, phantom, "="] & & B \\ & A \arrow[ur, "g"'] & \end{tikzcd} \quad\Longrightarrow\quad \begin{tikzcd} & & A \\ E \arrow[urr, bend left, "e"] \arrow[r, dashed, "h"'] & E \arrow[ur, "m"'] \arrow[u, phantom, "="] & \end{tikzcd}$$

  (Morphisms that have this property are called *regular monomorphisms*.) ⊿

◸**8.11.** A class of qbs-morphisms is *admissible* when, for every pullback square as follows, in which $m \in \mathcal{M}$ then necessarily $\overset{\bowtie}{\pi}_1 \in \mathcal{M}$:

$$\begin{tikzcd} f \bowtie m \arrow[r, "\overset{\bowtie}{\pi}_2"] \arrow[d, "\overset{\bowtie}{\pi}_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "m"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

Show that:

- Monomorphisms are admissible.
- Subspace embeddings are admissible.
- Borel embeddings are admissible. ⊿

◸**8.12.** Let $\mathcal{M}$ be an admissible class. An $\mathcal{M}$-*classifier* is a pair $\langle \Omega_{\mathcal{M}}, \top_{\mathcal{M}} \rangle$ consisting of:

- a space $\Omega_{\mathcal{M}}$; and
- an $\mathcal{M}$-morphism $\top_{\mathcal{M}} : \mathbb{1} \to \Omega_{\mathcal{M}}$

such that for every $\mathcal{M}$-morphism $m : X \to A$, there is a unique qbs morphism $\varphi : A \to \Omega_{\mathcal{M}}$ for which the following square is a pullback square:

$$\begin{tikzcd} X \arrow[r, "\langle\rangle"] \arrow[d, "m"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbb{1} \arrow[d, "\top_{\mathcal{M}}"] \\ A \arrow[r, dashed, "\varphi"'] & \Omega_{\mathcal{M}} \end{tikzcd}$$

In this case, we denote this unique φ by $[- \in m[X]]_{\mathcal{M}} : A \to \Omega_{\mathcal{M}}$.

Show:

- If $\mathcal{M}$ has a classifier in **Qbs**, then $\mathcal{M}$ contains only subspace embeddings.
- The indiscrete Booleans $\langle \llcorner \mathbb{2}_{\mathbf{Qbs}} \lrcorner, \underline{\mathbf{true}} \rangle$ form a subspace embedding classifier.
- The discrete Booleans $\langle \ulcorner \mathbb{2}^{\mathbf{Qbs}} \urcorner, \underline{\mathbf{true}} \rangle$ form a Borel embedding classifier.
- There are no monomorphism nor Borel monomorphism classifiers in **Qbs**. ⊿

A *factorisation system* $\langle \mathcal{E}, \mathcal{M} \rangle$ is a pair of classes of morphisms such that:

- $\mathcal{E}$ and $\mathcal{M}$ are closed under composition and contain all isomorphisms;
- every morphism $f : A \to B$ has an $\mathcal{E}$-$\mathcal{M}$ *factorisation*:

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "\mathcal{E} \ni e"'] & {} \arrow[d, phantom, "="] & B \\ & X \arrow[ur, "m \in \mathcal{M}"'] & \end{tikzcd}$$

- every morphism $m \in \mathcal{M}$ is right-orthogonal to every morphism $e \in \mathcal{E}$ (cf. Ex.8.10):

$$\begin{tikzcd} B \arrow[r, "e \in \mathcal{E}"] \arrow[d, "f"'] \arrow[dr, phantom, "="] & C \arrow[d, "g"] \\ S \arrow[r, "m \in \mathcal{M}"'] & A \end{tikzcd} \implies \begin{tikzcd} B \arrow[r, "e \in \mathcal{E}"] \arrow[d, "f"'] & C \arrow[d, "g"] \arrow[dl, dashed, "h"'] \\ S \arrow[r, "m \in \mathcal{M}"'] & A \end{tikzcd}$$

◸**8.13.** Show that $\langle \mathbf{epi}, \mathbf{subspace\ embedding} \rangle$ is a factorisation system. ⊿

◸**8.14.** A qbs morphism $e : A \to B$ is a *strong epimorphism* when the its action on random elements is surjective:

$$e \circ - : \mathcal{R}_A \twoheadrightarrow \mathcal{R}_B$$

Show that:

- The projection $\pi_1 : \mathbb{R}^2 \to \mathbb{R}$ is a strong epimorphism.
- Every strong epimorphism is surjective.
- Every map from a non-empty space into the terminal space $\langle\rangle : X \to \mathbb{1}$ is a strong epimorphism.

- If $f_i : A_i \to B_i$, $i \in I$, is a countable collection of strong epimorphisms, then their product $\prod_{i\in I} f_i : \prod_{i\in I} A_i \to \prod_{i\in I} B_i$ is a strong epimorphism. ⊿

◸**8.15.** Find an epimorphism that is not a strong epimorphism. ⊿

◸**8.16.** Show that $\langle \textbf{strong epimorphisms}, \textbf{mono} \rangle$ is a factorisation system. ⊿

References