## 8 Borel subspaces

*The* central notion in measure theory is that of a measurable subset — it is the defining concept of a measurable space. With quasi-Borel spaces, measurable subsets are a derived notion, but take a nonetheless central role.

 $\nabla$ 8.1. A measurable, or Borel, subset in a qbs A is a subset  $U \subseteq A_{\downarrow}$  such that the preimage under every random element  $\alpha \in \mathcal{R}_A$  is a Borel subset of the reals:  $\alpha^{-1}[U] \in \mathcal{B}$ . We denote by  $\mathcal{B}_A$  the set of Borel subsets of A.

Show that the measurable sets  $\mathcal{B}_A$  in a qbs A form a  $\sigma$ -algebra, and every random element is measurable w.r.t. this  $\sigma$ -algebra.

We denote the resulting measurable space by  $\begin{bmatrix} \mathbf{M}eas \\ A \end{bmatrix} := \begin{pmatrix} A \\ \mathbf{Set} \end{bmatrix}, \mathcal{B}_A \end{pmatrix}, \text{ and call it the free measurable space over } A.$ 

■ Show that  $U \subseteq A_j$  is measurable iff its indicator function  $[- \in U] : A \to 2^{Qbs}$  is a qbs morphism from A into the discrete qbs on the two-element set.

 $\bigtriangledown$  8.2. Find the Borel sets of the discrete qbs  $2^{2}$  and the indiscrete qbs  $2^{2}_{\text{Qbs}}$  on two elements. Generalise this result to the discrete and indiscrete qbses over any set X.

 $\bigtriangledown$  8.3. Show that the Borel subsets of  $\mathbb{R}$  in the standard sense coincide with the measurable subsets of the qbs  $\mathbb{R}$ .

 $\bigtriangledown$  8.4. Let A be a qbs and  $X \subseteq A$  be a subset.

Show that if  $U \subseteq A_{\downarrow}$  is Borel in A, then  $U \cap X$  is Borel in the subspace X:

 $U \in \mathcal{B}_A \implies U \cap X \in \mathcal{B}_X$ 

- Show that if X is itself a Borel subset, then  $\mathcal{B}_X \subseteq \mathcal{B}_A$ .
- $\blacksquare$  Show that the previous clause may fail if X is not Borel.

The Borel subsets of a subspace can be quite different from the Borel subsets of its superspace. For example, we may have a Borel subset  $V \in \mathcal{B}_X$  of the subspace that is not of the form  $U \cap X$  for any Borel subset  $U \in \mathcal{B}_A$  of the superspace.

Here's the intuition:

- A subset U in a qbs is measurable unless there is some random element that stops it from being measurable by mapping U onto a non-Borel inverse image.
- Wild' random elements may not factor through a subspace embedding  $X \rightarrow A$ .
- $\blacksquare$  So a subspace may have more Borel subsets in X than in its superspace.

If you want to see this intuition playing out, here is how to construct a counter-example:

 $\nabla 8.5$ . Let  $C_1 \subseteq \mathbb{R}$  be a non-Borel subset and  $C_2 \coloneqq \mathbb{R} \setminus C_1$  its complement, also non-Borel. Let  $\mathfrak{Z} \coloneqq \{0, 1, 2\}$  be a three-element set, and define two primitive random elements  $\alpha_i : \mathbb{R} \to \mathfrak{Z}$ :

$$\alpha_0 r \coloneqq \begin{cases} r \in C_1 : 0 \\ r \in C_2 : 2 \end{cases} \qquad \alpha_1 r \coloneqq \begin{cases} r \in C_1 : 1 \\ r \in C_2 : 2 \end{cases}$$

Take  $A \coloneqq \langle \Im, \operatorname{Cl}_{qbs} \{ \alpha_0, \alpha_1 \} \rangle$  to be the qbs over  $\Im$  with the smallest metaphorology (see Ex.7.9) containing  $\alpha_0$  and  $\alpha_1$ , and take  $X \coloneqq 2 \subseteq \Im$ .

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- Show that  $X, \{0\}, \{0, 2\} \notin \mathcal{B}_A$  are not Borel subsets in A.
- Show that if  $\alpha \in \mathcal{R}_A$  is a random element in A, then either  $\alpha$  is  $\sigma$ -simple or  $2 \in \text{Im}(\alpha)$ .

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Show that  $\{0\} \in \mathcal{B}_X$  is a Borel subset of the subspace X.

 $\nabla$ 8.6. Let  $f : A \to B$  be a qbs morphism. Show that:

- The inverse image under f restricts to a function  $\mathcal{B}_f : \mathcal{B}_B \to \mathcal{B}_A$ .
- $= \text{The underlying function } \begin{array}{c} f \\ \text{Set} \end{array} \text{ is a measurable function } \begin{array}{c} Meas \\ f \end{array} \text{ : } \begin{array}{c} Meas \\ A \end{array} \rightarrow \begin{array}{c} Meas \\ B \end{array} \text{.} \end{array} \begin{array}{c} \Delta \end{array}$

The collection of Borel sets has a universal property: it allows us to connect measurable spaces with quasi-Borel spaces as follows:

 $\nabla$ 8.7. For a measurable space M, define its set of random elements by  $\mathcal{R}_M \coloneqq \text{Meas}(\mathbb{R}, M)$ .

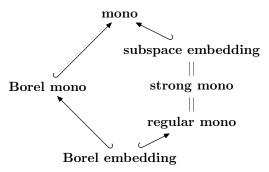
- Show that  $\mathcal{R}_M$  is a metaphorology, that is,  $M_{\mathbf{Qbs}} := \langle M_{\mathbf{Set}}, \mathcal{R}_M \rangle$  is a qbs. ■ For every measurable function  $f : M \to N$  between measurable spaces, show that its
- For every measurable function  $f: M \to N$  between measurable spaces, show that its underlying function is a qbs morphism  $f_{Qbs} : M_{Qbs} \to N_{Qbs}$ .
- Noticing that  $\__{\mathbf{Qbs}}$ : **Meas**  $\rightarrow$  **Qbs** is a (faithful) functor, show that it has a left adjoint equipping a qbs with its set of Borel subsets:  $\__{\mathbf{Qbs}}$ .

 $\bigtriangledown$  8.8. The free qbs functor  $\Box^{\mathbf{Qbs}}$ : Set  $\rightarrow$  Qbs doesn't preserve countable products.

This point is a natural place to stop, but if you're having fun with this material, then the rest of this sheet studies the relationships between natural notions of 'subspace'.

- $\blacksquare$   $m: A \Rightarrow B$  Monomorphisms: injective qbs morphisms.
- $m: A \hookrightarrow B$  Subspace embedding: injective on elements and surjective on randomelements that factor through the image.
- $m: A \rightarrowtail B$  Borel injections: monomorphisms whose image is a Borel subset.
- $\blacksquare$   $m: A \leftrightarrow B$  Borel embeddings: subspace embeddings whose image is a Borel subset.

We establish their following mutual relationships, where all inclusions are proper:



 $\bigtriangledown$  8.9. Place the following injections in the hierarchy of monomorphisms above:

The injection  $\top := \lambda \star .1 : [ \ 1 \ ] \to [ \ 2 \ ]$ . The injection  $\lambda x.x: [ \ 2 \ ] \to [ \ 2 \ ].$ 

- $= \text{ The injection } \lambda x.x: \begin{bmatrix} \mathbf{Qbs} \\ 2 \end{bmatrix} \mapsto \begin{bmatrix} 3 \\ \mathbf{Qbs} \end{bmatrix}.$
- The (subspace) inclusion  $\lambda x.x: C \hookrightarrow \mathbb{R}$  where C is a non-Borel subset of  $\mathbb{R}$ .
- $\nabla$  8.10. Let  $m: S \to A$  be a qbs morphism. Show that the following are equivalent:
- *m* is a subspace embedding, i.e.: there is a subset  $X \subseteq [A]$  and an isomorphism  $m' : B \xrightarrow{\cong} X$  satisfying:

$$\begin{array}{c} S & m \\ m' & \cong \\ X & \lambda x.x \end{array} A$$

■ *m* is *right-orthogonal* to every empimorphism  $e: B \twoheadrightarrow C$ : for every commuting square as on the left, there is a unique morphism  $h: C \to S$  commuting the triangles on the right:

$$B \xrightarrow{e} C$$

$$f \downarrow = \downarrow g$$

$$S \xrightarrow{m} A$$

$$B \xrightarrow{e} C$$

$$f \downarrow = h_{----} \downarrow g$$

$$S \xleftarrow{m} A$$

(Morphisms that have this property are called *strong monomorphisms*.) m is an *equaliser* of some parallel pair of morphisms  $f, g: A \rightarrow B$ :

 $\blacksquare$  *m* equalises *f* and *g*:

$$S \xrightarrow{m} A \xrightarrow{f} B$$

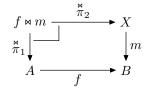
– and every equalising morphism  $e: C \to A$  factors uniquely through m:

$$E \xrightarrow{e}_{A} \xrightarrow{f}_{B} \xrightarrow{e}_{E} \xrightarrow{e}_{m} A$$

(Morphisms that have this property are called *regular monomorphisms*.)

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 $\nabla$  8.11. A class of qbs-morphisms is *admissible* when, for every pullback square as follows, in which  $m \in \mathcal{M}$  then necessarily  $\overset{\bowtie}{\pi}_1 \in \mathcal{M}$ :



Show that:

- Monomorphisms are admissible.
- **—** Subspace embeddings are admissible.
- Borel embeddings are admissible.

**Exercises** 

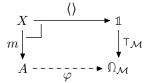
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 $\nabla$ 8.12. Let  $\mathcal{M}$  be an admissible class. An  $\mathcal{M}$ -classifier is a pair  $\langle \Omega_{\mathcal{M}}, T_{\mathcal{M}} \rangle$  consisting of:

- $\blacksquare$  a space  $\Omega_{\mathcal{M}}$ ; and

such that for every  $\mathcal{M}$ -morphism  $m: X \to A$ , there is a unique qbs morphism  $\varphi: A \to \Omega_{\mathcal{M}}$  for which the following square is a pullback square:



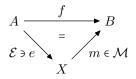
In this case, we denote this unique  $\varphi$  by  $[- \in m[X]]_{\mathcal{M}} : A \to \Omega_{\mathcal{M}}$ . Show:

- **—** If  $\mathcal{M}$  has a classifier in **Qbs**, then  $\mathcal{M}$  contains only subspace embeddings.
- The indiscrete Booleans  $\langle 2_{Qbs}, \underline{true} \rangle$  form a subspace embedding classifier.
- The discrete Booleans  $\langle 2^{, \mathbf{Qbs}}, \underline{\mathbf{true}} \rangle$  form a Borel embedding classifier.
- There are no monomorphism nor Borel monomorphism classifiers in **Qbs**.

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A factorisation system  $\langle \mathcal{E}, \mathcal{M} \rangle$  is a pair of classes of morphisms such that:

- $\mathbf{z}$  and  $\mathcal{M}$  are closed under composition and contain all isomorphisms;
- every morphism  $f: A \to B$  has an  $\mathcal{E}$ - $\mathcal{M}$  factorisation:



every morphism  $m \in \mathcal{M}$  is right-orthogonal to every morphism  $e \in \mathcal{E}$  (cf. Ex.8.10):



 $\bigtriangledown$  8.13. Show that (epi, subspace embedding) is a factorisation system.

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 $\bigtriangledown$  8.14. A qbs morphism  $e: A \rightarrow B$  is a *strong epimorphism* when the its action on random elements is surjective:

 $e \circ - : \mathcal{R}_A \twoheadrightarrow \mathcal{R}_B$ 

Show that:

- The projection  $\pi_1 : \mathbb{R}^2 \to \mathbb{R}$  is a strong epimorphism.
- Every strong epimorphism is surjective.
- Every map from a non-empty space into the terminal space  $\langle \rangle : X \to 1$  is a strong epimorphism.

## REFERENCES

If $f_i: A_i \to B_i$ , $i \in I$ , is a countable collection of strong epimorphisms, then their	product
$\prod_{i \in I} f_i : \prod_{i \in I} A_i \to \prod_{i \in I} B_i$ is a strong epimorphism.	Δ
$\bigtriangledown$ 8.15. Find an epimorphism that is not a strong epimorphism.	Δ
$\bigtriangledown$ 8.16. Show that (strong epimorphisms, mono) is a factorisation system.	Δ

## References