7 Qbs constructions

In this sheet you'll construct new qbses out of given ones. If the development starts to feel too abstract, skip to the next sheet and come back to it when needed.

◸**7.1.** Let A be a qbs and $X \subseteq \llcorner A \lrcorner$ a set of points. We can equip X with a qbs structure by taking as random elements all the random elements of A whose image is in X:

$$\mathcal{R}_X := \{\alpha : \mathbb{R} \to X | \alpha \in \mathcal{R}_A\}$$

This qbs is called the *subspace of A induced by X*.

- Check that the subspace X is a qbs, and that the inclusion $X \hookrightarrow \llcorner A \lrcorner$ is a qbs morphism.

A *subspace embedding* $m : B \hookrightarrow A$ of a qbs B into A is a qbs morphism $m : B \to A$ where:

- $\llcorner m \lrcorner : \llcorner B \lrcorner \to \llcorner A \lrcorner$ is injective; and
- $\llcorner m \lrcorner \circ : \mathcal{R}_B \to \mathcal{R}_A$ is surjective.

Show the following:

- Every point $\underline{x} := \lambda \star . x : \mathbb{1} \to A$ is a subspace embedding.
- Each inclusion $X \subseteq \llcorner A \lrcorner$ of a subspace X into its superspace A is a subspace embedding.
- Not every monomorphism in **Qbs** is a subspace embedding.
- Every isomorphism is a subspace embedding.
- Every qbs morphism factors as the composition $m \circ e$ of an epimorphism followed by a subspace embedding. ◿

◸**7.2.** Let A be a qbs, X a set, and $g : X \to \llcorner A \lrcorner$ any function from X into the points of A. Show that g carries a qbs morphism $f : \ulcorner X \urcorner_{\mathbf{Qbs}} \to A$ from the indiscrete space over X into A, i.e., $\llcorner f \lrcorner = g$, iff the subspace $g[X] \hookrightarrow A$ is indiscrete.

Deduce that every morphism $f : \ulcorner X \urcorner_{\mathbf{Qbs}} \to \mathbb{R}$ is constant. ◿

◸**7.3.** Find the terminal qbs $\mathbb{1}$ and the initial qbs $\mathbb{0}$. ◿

◸**7.4.** Let X_1, X_2 be qbses. Construct their *product* $\langle X_1 \times X_2, \pi_1, \pi_2 \rangle$:

- The set of points is the cartesian product: $\llcorner X_1 \times X_2 \lrcorner := \llcorner X_1 \lrcorner \times \llcorner X_2 \lrcorner$.
- The random elements are tupling of random elements:

  $$\mathcal{R}_{X_1 \times X_2} := \{\alpha : \mathbb{R} \to X_1 \times X_2 | \pi_1 \circ \alpha \in \mathcal{R}_{X_1}, \pi_2 \circ \alpha \in \mathcal{R}_{X_2}\}$$

  We can think of random elements in $X_1 \times X_2$ as correlated random-elements in the product space.
- The two projections are given by the set-theoretic projections:

  $$\pi_i : X_1 \times X_2 \to X_i$$
  $$\pi_i \langle x_1, x_2 \rangle := x_i$$

Show:

- $X_1 \times X_2$ satisfies the qbs axioms.

- Each projection $\pi_i : X_1 \times X_2 \to X_i$ is a qbs morphism.
- The universal property of the product (see Ex.2.15).
- Generalise: construct the product $\prod_{i \in I} X_i$ of any I-indexed family of spaces. $\lrcorner$

7.5. Let X_1, X_2 be qbses. Construct their *coproduct / disjoint union* $\langle X_1 \amalg X_2, \iota_1, \iota_2 \rangle$:

- The set of points is the disjoint union: $\llcorner X_1 \amalg X_2 \lrcorner := \llcorner X_1 \lrcorner \amalg \llcorner X_2 \lrcorner := (\{1\} \times X_1) \cup (\{2\} \times X_2)$.
- The random elements are binary recombinations of random elements:

$$\mathcal{R}_{X_1 \amalg X_2} := \left\{ \alpha : \mathbb{R} \to X_1 \amalg X_2 \;\middle|\; \begin{aligned} &\exists \alpha_1 \in \mathcal{R}_{X_1}, \alpha_2 \in \mathcal{R}_{X_2}. \alpha = [\alpha_1, \alpha_2] \\ &\qquad := \lambda x. \begin{cases} x = \iota_1 x_1 : & \alpha_1\, x_1 \\ x = \iota_2 x_2 : & \alpha_2\, x_2 \end{cases} \end{aligned} \right\}$$

  We can think of random elements in $X_1 \amalg X_2$ as splitting the probability between the two spaces.
- The two injections are given by the set-theoretic injections:

$$\iota_i : X_i \to X_1 \amalg X_2$$
$$\iota_i x := \iota_i x := \langle i, x \rangle$$

Show:

- $X_1 \amalg X_2$ satisfies the qbs axioms.
- Each injection $\iota_i : X_i \to X_1 \amalg X_2$ is a qbs morphism.
- The universal property of the coproduct (see Ex.4.3).
- Generalise: construct the coproduct $\coprod_{i \in I} X_i$ of any I-indexed family of spaces. $\lrcorner$

7.6. Let $X_1 \xrightarrow{f_1} Y \xleftarrow{f_2} X_2$ be two qbs morphisms. Construct their *pullback* $\left\langle f_1 \bowtie f_2, \overset{\bowtie}{\pi}_1, \overset{\bowtie}{\pi}_2 \right\rangle$:

- The set of points is the set-theoretic pullback:

$$\llcorner f_1 \bowtie f_2 \lrcorner := \left\{ \langle x_1, x_2 \rangle \in \llcorner X_1 \lrcorner \times \llcorner X_2 \lrcorner \;\middle|\; f_1\, x_1 = f_2\, x_2 \right\}$$

- The random elements are tupling of random elements whose projections agree:

$$\mathcal{R}_{f_1 \bowtie f_2} := \left\{ \alpha : \mathbb{R} \to \llcorner f_1 \bowtie f_2 \lrcorner \;\middle|\; \overset{\bowtie}{\pi}_i \circ \alpha \in \mathcal{R}_{X_i}, i = 1, 2 \right\}$$

- The projections are given by the set-theoretic projections:

$$\overset{\bowtie}{\pi}_i : X_1 \times X_2 \to X_i$$
$$\overset{\bowtie}{\pi}_i \langle x_1, x_2 \rangle := x_i$$

Show:

- $f_1 \bowtie f_2$ satisfies the qbs axioms.
- Each projection $\overset{\bowtie}{\pi}_i : f_1 \bowtie f_2 \to X_i$ is a qbs morphism.
- The universal property of the pullback:

$$\begin{tikzcd} f_1 \bowtie f_2 \arrow[r, "\overset{\bowtie}{\pi}_2"] \arrow[d, "\overset{\bowtie}{\pi}_1"'] \arrow[dr, phantom, "="] & X_2 \arrow[d, "f_2"] \\ X_1 \arrow[r, "f_1"'] & Y \end{tikzcd} \quad \text{and every} \quad \begin{tikzcd} Z \arrow[drr, bend left, "p_2"] \arrow[ddr, bend right, "p_1"'] & & \\ & \arrow[r, phantom, "="] & X_2 \arrow[d, "f_2"] \\ & X_1 \arrow[r, "f_1"'] & Y \end{tikzcd} \implies \begin{tikzcd} Z \arrow[dr, dashed, "h"] \arrow[drr, bend left, "p_2"] \arrow[ddr, bend right, "p_1"'] & & \\ & f_1 \bowtie f_2 \arrow[r, "\overset{\bowtie}{\pi}_2"'] \arrow[d, "\overset{\bowtie}{\pi}_1"] & X_2 \\ & X_1 & \end{tikzcd}$$

- The pullback is a subspace of the product: $f_1 \bowtie f_2 \hookrightarrow X_1 \times X_2$. $\triangle$

$\triangledown$**7.7.** The pullback of a subspace embedding $m : S \hookrightarrow Y$ along any morphism $f : X \to Y$ is a subspace embedding $\overset{\bowtie}{\pi}_1 : f \bowtie m \hookrightarrow X$. $\triangle$

We can summarise this situation in the previous exercise in this diagram, where the right-angle marker inside the square marks it as a pullback:

$$\begin{tikzcd} f_1 \bowtie f_2 \arrow[r, "\overset{\bowtie}{\pi}_2"] \arrow[d, hook, "\overset{\bowtie}{\pi}_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & S \arrow[d, hook, "m"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

$\triangledown$**7.8.** Show:

- The free qbs functor $\ulcorner - \urcorner^{\mathbf{Qbs}} : \mathbf{Set} \to \mathbf{Qbs}$ preserves finite products.
- The free qbs functor $\ulcorner - \urcorner^{\mathbf{Qbs}} : \mathbf{Set} \to \mathbf{Qbs}$ doesn't preserve products.
- The indiscrete qbs functor $\llcorner - \lrcorner_{\mathbf{Qbs}} : \mathbf{Set} \to \mathbf{Qbs}$ doesn't preserve finite coproducts.
- Deduce that the sequence of adjunctions: $\ulcorner - \urcorner^{\mathbf{Qbs}} \dashv \llcorner - \lrcorner_{\mathbf{Set}} \dashv \llcorner - \lrcorner_{\mathbf{Qbs}}$ between **Qbs** and **Set** doesn't have a further left adjoint nor a further right adjoint. $\triangle$

Let X be a set. A *metaphorology*[1] over X is a set $\mathcal{R} \subseteq X^{\mathbb{R}}$ of functions from $\mathbb{R}$ to X that satisfies the qbs axioms (contains all constant functions and closed and measurable pre-composition and recombination). Thus, a qbs A is a set $\llcorner A \lrcorner$ equipped with a metaphorology $\mathcal{R}_A$. (In Ex.6.3 we called this concept a qbs structure.)

$\triangledown$**7.9.** Let X be a set and $E \subseteq X^{\mathbb{R}}$ any set of functions. Show that the smallest metaphorology $\mathrm{Cl}_{\mathrm{qbs}}E$ on X containing E is given by the recombinations of measurable pre-compositions of E-elements and constant functions:

$$\mathrm{Cl}_{\mathrm{qbs}}E = \left\{ [\lambda r \in U_i.\alpha_i(r)]_{i \in I} \;\middle|\; \begin{array}{l} I \text{ is countable}, \mathbb{R} = \biguplus_{i \in I} U_i, \text{ and for every } i \in I\text{:} \\ U_i \in \mathcal{B}_{\mathbb{R}}, \text{and} \\ \text{either } \alpha_i \text{ constant, or there is some} \\ \text{measurable } \varphi_i : U_i \to \mathbb{R},\ \beta_i \in E \text{ s.t.: } \alpha_i = \beta_i \circ \varphi_i \end{array} \right\} \quad \triangle$$

$\triangledown$**7.10.** Show that the functor $\llcorner - \lrcorner_{\mathbf{Set}} : \mathbf{Qbs} \to \mathbf{Set}$ generates limits and colimits, but doesn't create limits nor colimits (see discussion before Ex.3.27). Deduce that **Qbs** is complete and cocomplete. $\triangle$

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[1] I'm open to suggestions for other names. Going back to its original roots, 'metaphor' originates from the Greek $\mu\varepsilon\tau\alpha$ ('meta', across) and $\varphi\varepsilon\rho\omega$ ('phero', to carry). This choice makes 'metaphors' an appealing alternative to 'random element'. The other candidate was 'stochastology'.

References