2 Measurable spaces and functions

Try these exercises if you're new to measure theory and are curious about it.

◸**2.1.** Show that each subset is Borel in each measurable space:

- The diagonal $\left\{\langle r, r\rangle \in \mathbb{R}^2 \middle| r \in \mathbb{R}\right\}$ in the Euclidean plane $\mathbb{R}^2$.
- The 3-dimensional open ball $\left\{\langle x, y, z\rangle \in \mathbb{R}^3 \middle| x^2 + y^2 + z^2 < 1\right\}$ in the Euclidean space $\mathbb{R}^3$.
- The 2-dimensional sphere $\left\{\langle x, y, z\rangle \in \mathbb{R}^3 \middle| x^2 + y^2 + z^2 = 1\right\}$ in the Euclidean space $\mathbb{R}^3$.

If you're unsure how to approach the exercise, try the rest of this section first. ◿

◸**2.2.** Prove that the following functions over $\mathbb{R}$ are measurable, for all $r \in \mathbb{R}$:

- $(r+) \coloneqq \lambda s.r + s$
- $(r\cdot) \coloneqq \lambda s.r \cdot s$ ◿

We can organise measurable spaces and functions into a category called **Meas**: the measurable spaces are the objects and the measurable functions are the morphisms between these objects. You already know another category: **Set**. Its objects are sets and its morphisms are functions between those sets. This course isn't about category theory, but we will take advantage of category theory to help us relate concepts that live in different areas of mathematics. So if you never worked with categories before, you can use this course to learn a bit more about categories. In that case, please take full advantage of myself and your categorically-savvy course-mates!

If you are such a categorically-savvy person, you already covered the next few exercises in the past and may want to skip to Ex.2.8.

◸**2.3.** Let's spell out the category structure of **Meas**:

- Objects are measurable spaces X, Y;
- Morphisms $f : X \to Y$ are measurable functions of the same type.
- Identities $\mathrm{id}_X : X \to X$ are the identity functions $\lambda x.x$ of the same type.
- The composition of $f : Y \to Z$ and $g : X \to Y$ is the composed function $f \circ g : X \to Z$.

Show the implicit statements in the last two clauses:

- The identity function is a measurable functions $\mathrm{id}_X : X \to X$.
- The composition is a measurable function $f \circ g : X \to Z$. ◿

Having spelled out the structure, we should now check that this structure is a category:

◸**2.4.** Show that:

- identities are neutral w.r.t. composition — for all $f : X \to Y$: $f \circ \mathrm{id}_X = f = \mathrm{id}_Y \circ f$; and
- composition is associative: $f \circ (g \circ h) = (f \circ g) \circ h$. ◿

You may have found 1-line proofs for each of the category axioms for **Meas**. This may feels silly and tedious. It also usually means there's a structural reason why those proofs work. Here is one. There is a functor $\llcorner - \lrcorner : \mathbf{Meas} \to \mathbf{Set}$, that is, there is an assignment:

- to each measurable space X, we assign set its set of points $\llcorner X \lrcorner$;
- to each measurable function $f : X \to Y$, we assign its corresponding function between the corresponding sets of points $f : \llcorner X \lrcorner \to \llcorner Y \lrcorner$;

and this assignment respects identities and composition:

◸**2.5.** Show that:

- $\llcorner \mathrm{id}_X \lrcorner = \mathrm{id}_{\llcorner X \lrcorner}$ for every measurable space X; and
- $\llcorner f \circ g \lrcorner = \llcorner f \lrcorner \circ \llcorner g \lrcorner$ for every pair of composable measurable functions $X \xrightarrow{g} Y \xrightarrow{f} Z$. ◿

Equations between functions (and more generally, morphisms) leave the intermediate spaces implicit in the background. We can mention both the spaces and the last two equations diagrammatically:

$$\begin{tikzcd} X \arrow[d, bend right, "\llcorner \mathrm{id}_X \lrcorner"'] \arrow[d, bend left, "\mathrm{id}_{\llcorner X \lrcorner}"] \arrow[d, phantom, "="] \\ X \end{tikzcd} \qquad \begin{tikzcd} X \arrow[dr, "\llcorner f \lrcorner"] \arrow[dd, "\llcorner g \circ f \lrcorner"'] \arrow[ddr, phantom, "="] & \\ & Y \arrow[dl, "\llcorner g \lrcorner"] \\ Z & \end{tikzcd}$$

Vertices in the diagrams are objects, and directed edges are labelled by morphisms between these objects. We'll use a stretched equality notation to mark edges labelled by identity morphisms, and often omit the actual label. Each face has a source and a sink, and two paths from the source to the sink comprising of composable morphisms. The equality sign on a face states an equality between the composion of the morphisms on the two paths around the face. In the left diagram, it means $\llcorner \mathrm{id}_X \lrcorner = \mathrm{id}_{\llcorner X \lrcorner}$ and on the right diagram, it means $\llcorner f \circ g \lrcorner = \llcorner f \lrcorner \circ \llcorner g \lrcorner$.

Let $\mathcal{B}$ and $\mathcal{C}$ be category *structures*, so they have objects, morphisms, identities, and composition operators, but we make no assumptions that identities are neutral or composition is associative. A functor $F : \mathcal{B} \to \mathcal{C}$ is *faithful* when, for every pair of morphisms of the same type $f, g : X \to Y$ in $\mathcal{B}$, we have: $Ff = Fg \implies f = g$. So the functorial action on morphisms is injective.

◸**2.6.** Prove:

- The functor $\llcorner - \lrcorner : \mathbf{Meas} \to \mathbf{Set}$ is faithful.
- Faithful functors *reflect* categories: if $F : \mathcal{B} \to \mathcal{C}$ is faithful and $\mathcal{C}$ is a category, then $\mathcal{B}$ is also a category.
- Deduce that **Meas** is a category. ◿

This kind of 'short-cut' is not a short-cut at all: we replaced 3×1-line proofs with the same 3×1-line proofs, merely done abstractly, *and* had to prove $\llcorner - \lrcorner$ is a functor, which involves 2 additional proofs.

My answer, and it may not be *your* answer, is that being able to relate concepts in **Meas** and **Set** and how to transfer properties (like being a category) across these relationships is a useful technique, and it's worth learning. Here are a few more simple examples:

◸**2.7.** A morphism $f : X \to Y$ in a category $\mathcal{C}$ is an *isomorphism* when there is a morphism $g : Y \to X$ such that $f \circ g = \mathrm{id}_Y$ and $g \circ f = \mathrm{id}_X$:

$$\begin{tikzcd} X \arrow[rr, equal] \arrow[dr, "f"'] & & X \arrow[dr, "f"] & \\ & Y \arrow[ur, "g"] \arrow[rr, equal] \arrow[ur, phantom, "="] & & Y \end{tikzcd}$$

Show the following:

- Every functor $F : \mathcal{B} \to \mathcal{C}$ preserves isomorphisms: if $f : X \to Y$ is an isomorphism in $\mathcal{B}$, then $Ff : FX \to FY$ is an isomorphism in $\mathcal{C}$.
- Faithful functors reflect isomorphism *pairs*: for all $X \xrightarrow{f} Y \xrightarrow{g} X$ in $\mathcal{B}$, if Ff and Fg are each others' inverses in $\mathcal{C}$ then f and g are each others' inverses in $\mathcal{B}$.
- The faithful functor $\llcorner - \lrcorner : \mathbf{Meas} \to \mathbf{Set}$ does not reflect isomorphisms: there is a measurable function $f : X \to Y$ that is not an isomorphism, but its underlying function $f : \llcorner X \lrcorner \to \llcorner Y \lrcorner$ is bijective. ◿

Like any formalism, categories takes practice to pick the vocabulary up and to use it effectively, for example, only when it's needed. In the rest of the course, every statement involving categories will be accompanied by its non-categorical formulation, or may be safely skipped. Whether or not you choose to use the language of categories is up to you. At the very least, these statements offer another source of exercise for you.

◸**2.8.** Let V be a measurable space and $A \subseteq \llcorner V \lrcorner$ any subset.

- Prove that $\mathcal{B}_A \coloneqq \{U \cap A | U \in \mathcal{B}_V\}$ is a σ-algebra.
- Prove that if A is measurable, i.e., $A \in \mathcal{B}_V$, then $\mathcal{B}_A = \{U \in \mathcal{B}_V | U \subseteq A\}$.
- Show that the inclusion is a measurable function:

$$\begin{aligned} &i : A \subseteq V \\ &ix \coloneqq x \end{aligned}$$

- Let $f : V \to W$ be a measurable function then the restriction of f to A is measurable:

$$\begin{aligned} &f|_A : A \to W \\ &f|_A\, x \coloneqq fx \end{aligned}$$

◿

◸**2.9.** Prove that the following functions are measurable:

- $(\frac{1}{-}) : \mathbb{R}_{\neq 0} \to \mathbb{R}_{\neq 0}$ where $\mathbb{R}_{\neq 0} \coloneqq \mathbb{R} \setminus \{0\}$.
- $|-| : \mathbb{R} \to \mathbb{R}_{\geq 0}$ where $\mathbb{R}_{\geq 0} \coloneqq [0, \infty)$. ◿

◸**2.10.** Show that the inclusion $i : A \subseteq V$ is *cartesian* in the following way: for every measurable space W and measurable function $f : W \to V$ such that $\mathrm{Im}(f) \subseteq A$ there is a unique measurable function $h : W \to A$ with $f = i \circ h$:

$$\begin{tikzcd} \llcorner W \lrcorner \arrow[dr, "u"'] \arrow[drr, bend left, "\llcorner f \lrcorner"] \arrow[drr, phantom, "="] & & \\ & \llcorner A \lrcorner \arrow[r, "\llcorner i \lrcorner"'] & \llcorner V \lrcorner \end{tikzcd} \implies \begin{tikzcd} W \arrow[dr, dashed, "h"'] \arrow[drr, bend left, "f"] \arrow[drr, phantom, "="] & & \\ & A \arrow[r, "i"'] & V \end{tikzcd}$$

in **Set** in **Meas** ◿

◸**2.11.** If you enjoyed the previous exercise, try generalising it. Let V be a measurable space, A a set, and $v : A \to \llcorner V \lrcorner$ a function. Show that v has a *cartesian lifting*:

- a measurable space $v^*V \in \mathbf{Meas}$, together with

- a measurable function $\dot{v} : v^*V \to V$,

such that:

- $\llcorner v^*V \lrcorner = A$ and $\llcorner \dot{v} \lrcorner = v$; and
- for every measurable function $f : W \to V$ and function $u : \llcorner W \lrcorner \to \llcorner A \lrcorner$, if the equation on the left holds then there is a unique measurable function $h : W \to v^*A$ satisfying $\llcorner h \lrcorner = u$ and the eqution on the right:

$$\begin{tikzcd} \llcorner W \lrcorner \arrow[dr, "u"'] \arrow[drr, bend left, "\llcorner f \lrcorner"] \arrow[drr, phantom, "="] & & \\ & A \arrow[r, "v"'] & \llcorner V \lrcorner \end{tikzcd} \implies \begin{tikzcd} W \arrow[dr, dashed, "h"'] \arrow[drr, bend left, "f"] \arrow[drr, phantom, "="] & & \\ & v^*V \arrow[r, "\dot{v}"'] & V \end{tikzcd}$$

in **Set** in **Meas**

This fact states that the functor $\llcorner - \lrcorner : \mathbf{Meas} \to \mathbf{Set}$ is a Grothendieck fibration. We won't use this fact directly in the sequel. ◿

◸**2.12.** Let X be a set.

- Prove that every intersection of σ-algebras over X is a σ-algebra over X.

Let $\mathcal{U} \subseteq \wp A$ be a family of subsets of A. The σ-algebra $\sigma(\mathcal{U})$ *generated* by $\mathcal{U}$ is the smallest σ-algebra containing $\mathcal{U}$:

$$\sigma(\mathcal{U}) \coloneqq \bigcap \{\mathcal{B} \subseteq \wp X | \mathcal{B} \text{ is a } \sigma\text{-algebra and } \mathcal{U} \subseteq \mathcal{B}\}$$

Prove:

- If $\mathcal{U} \subseteq \mathcal{V}$ then $\sigma(\mathcal{U}) \subseteq \sigma(\mathcal{V})$.
- If $\mathcal{U}$ is already a σ-algebra, then $\sigma(\mathcal{U}) = \mathcal{U}$.
- Let V be a measurable space and $f : \llcorner V \lrcorner \to X$ any function. Prove that $f : V \to \langle X, \sigma(\mathcal{U}) \rangle$ is measurable iff for every $A \in \mathcal{U}$, we have $f^{-1}[U] \in \mathcal{B}_V$. ◿

◸**2.13.** Let X be a set, and set $\{[X]\} \coloneqq \left\{U \subseteq X \middle| U \text{ is countable or } U^{\complement} \text{ is countable}\right\} \subseteq \wp X$.

- Show that $\{[X]\}$ is a σ-algebra over X. This σ-algebra is known as the countable-co-countable σ-algebra.
- Show that $\{[X]\} = \sigma(\{\{x\} | x \in X\})$ is the σ-algebra generated by the singletons. ◿

If you know some transfinite induction, you might want a predicative definition of $\sigma(\mathcal{U})$. In that case, have a look at the (extensive!) bunch of exercises in Sec. A.

◸**2.14.** Let $A \subseteq \llcorner V \lrcorner$ be a subset of a measurable space V. Show that if $\mathcal{U}$ generates the σ-algebra of V, then $\mathcal{U}' \coloneqq A \cap [\mathcal{U}]$ generates the σ-algebra of the subspace A. ◿

◸**2.15.** Given families of subsets $\mathcal{U} \subseteq \wp X$ and $\mathcal{V} \subseteq \wp Y$, define their *box* σ-algebra:

$$\mathcal{U} \otimes \mathcal{V} \coloneqq \sigma \{A \times B | A \in \mathcal{U}, B \in \mathcal{V}\}$$

Let U, V be two measurable spaces.

- Set $U \times V := \left\langle \llcorner U \lrcorner \times \llcorner V \lrcorner, \mathcal{B}_U \otimes \mathcal{B}_V \right\rangle$, and show that the cartesian projections $\pi_1 : U \times V \to U$ and $\pi_2 : U \times V \to V$ are measurable.
- Show that $\langle U \times V, \pi_1, \pi_2 \rangle$ is the categorical product: for every measurable space W and pair of measurable functions $f : W \to U$ and $g : W \to V$, there is a unique measurable function $\langle f, g \rangle : W \to U \times V$ such that:

$$\begin{tikzcd} & W \arrow[dl, bend right, "f"'] \arrow[d, dashed, "\langle f, g \rangle"] \arrow[dr, bend left, "g"] & \\ U & U \times V \arrow[l, "\pi_1"] \arrow[r, "\pi_2"'] & V \end{tikzcd}$$

- Show that if $\mathcal{U}$ generates $\mathcal{B}_U$ and $\mathcal{V}$ generates $\mathcal{B}_V$, then $\mathcal{U} \otimes \mathcal{V} = \mathcal{B}_{U \times V}$. ◿

◸**2.16.** Let $\vec{V} = \langle V_i \rangle_{i \in I}$ be an I-indexed family of measurable spaces.

- Find their categorical product $\prod_{i \in I} V_i$.
- Find an example family and an I-indexed family of measurable subsets $A_i \in \mathcal{B}_{V_i}$ so that the cartesian product $\prod_{i \in I} A_i$ is not a Borel set in the categorical product $\prod_{i \in I} V_i$. ◿

◸**2.17.** A measurable space $\mathbb{0}$ is *initial* when there is a unique measurable function $[\,] : \mathbb{0} \to V$ for every measurable space V. Similarly, a measurable space $\mathbb{1}$ is *terminal* when there is a unique measurable function $\langle\rangle : V \to \mathbb{1}$ for every measurable space V.

(These concepts make sense in every category.)

- Show that **Meas** has exactly one initial space.
- Show that **Meas** has multiple terminal spaces.
- Show that terminal spaces are the product of an empty family of spaces. ◿

◸**2.18.** Show that each family of subsets generates the σ-algebra of the given space:

- $\{(-\infty, a) | a \in \mathbb{R}\}$, $\{(-\infty, a] | a \in \mathbb{R}\}$, $\{(-\infty, q) | q \in \mathbb{Q}\}$, $\{[a, b) | a, b \in \mathbb{R}\}$ all generate $\mathcal{B}_{\mathbb{R}}$.
- $\{C \cap I_k^n | n \in \mathbb{N}, k \in \mathbf{Fin}\, 2^n\}$ generate the Borel sets of the Cantor space $\mathbb{C}$. What are the corresponding subsets of $\mathbb{T} := 2^{\mathbb{N}}$?
- Show that the set of hemispheres generates the Borel sets of the unit 2-sphere. ◿

◸**2.19.** Let A be a set. The powerset $\wp A$ is a σ-algebra on A (why?). Define the *discrete* measurable space over A by $\ulcorner A \urcorner := \langle A, \wp A \rangle$. Show:

- For every measurable space V, each function $f : A \to \llcorner V \lrcorner$ is in fact a measurable function $f : \ulcorner A \urcorner \to V$.

The set $\{\varnothing, A\}$ is also a σ-algebra on A (why?). Define the *indiscrete* measurable space over A by $\llcorner A \lrcorner_{\mathbf{Meas}} := \langle A, \{\varnothing, A\} \rangle$. Show:

- For every measurable space V, each function $f : \llcorner V \lrcorner \to A$ is in fact a measurable function $f : V \to \llcorner A \lrcorner_{\mathbf{Meas}}$. ◿

◸**2.20.** Let A be a set and V a measurable space.

- Given a function $f : B \to A$, a subset $X \subseteq B$ is *f-saturated* when $x \in X$ and $fx = fy$ imply $y \in X$. Show that the f-saturated sets form a *topology*:
  - The empty $\varnothing$ set is f-saturated;
  - Arbitrary unions of f-saturated sets are f-saturated; and
  - Finite intersections of f-saturated sets are f-saturated.
    (In fact, arbitrary intersections of f-saturated sets are f-saturated.)
- Show that $f : V \to \ulcorner A \urcorner$ is measurable iff every f-saturated set is measurable.
- Let B be a set and $X_0 \subseteq B$ a subset. We say that a subset $X \subseteq B$ is *X_0-atomic* when $X_0 \subseteq X$ or $X_0 \cap X = \varnothing$. Show that the X_0-atomic subsets are a topology: the empty set is atomic, and finite intersections and arbitrary unions of atomic subsets are atomic.
- Show that $f : \llcorner \underset{\mathbf{Meas}}{A} \lrcorner \to V$ is measurable iff all the measurable subsets in V are $\mathrm{Im}(f)$-atomic. ◿

◸**2.21.** Let $\ulcorner \mathbb{R} \urcorner := \langle \mathbb{R}, \wp\mathbb{R} \rangle$ be the discrete measurable space over $\mathbb{R}$, and $\tilde{\mathbb{R}}$ be the countable-cocountable measurable space over $\mathbb{R}$. We'll show that the diagonal $\{\langle r, r \rangle | r \in \mathbb{R}\} \subseteq \wp(\mathbb{R} \times \mathbb{R})$ is not a measurable subset of $\ulcorner \mathbb{R} \urcorner \times \tilde{\mathbb{R}}$.

Define the following predicate $\Phi \subseteq \wp(\mathbb{R} \times \mathbb{R})$. Given $K \subseteq \mathbb{R} \times \mathbb{R}$, then $\Phi(K)$ holds when there is a countable sequence of real numbers $\vec{b} \in \mathbb{R}^{\mathbb{N}}$ such that for every $x \in \mathbb{R}$, if there is *some* $y_0 \notin \{b_n | n \in \mathbb{N}\}$ with $\langle x, y_0 \rangle \in K$, then *for all* $y \notin \{b_n | n \in \mathbb{N}\}$ we have $\langle x, y \rangle \in K$.

The intuition behind Φ: there is a countable collection of equality constraints on the second component we need to check in order to decide whether a pair is in K. Prove the following.

- $\Phi(K)$ iff there is some $\vec{b} \in \mathbb{R}^{\mathbb{N}}$ and a function $\varphi : \mathbb{R} \to 2^{\mathbb{N}+1}$ such that the indicator function of K is given by:

$$[(x, y) \in K] = \begin{cases} \exists n. y = b_n, \varphi(x, \iota_1 n) = \mathbf{true} : & \mathbf{true} \\ \text{otherwise:} & \varphi(x, \iota_2 \star) \end{cases}$$

- The diagonal $\{\langle r, r \rangle \in \mathbb{R} \times \mathbb{R} | r \in \mathbb{R}\}$ is not in Φ.
- $\Phi(A \times B)$ for every countable and cocountable subset $B \subseteq \mathbb{R}$.
- Φ is closed under countable unions and countable intersections.
- For every measurable subset $K \in \mathcal{B}_{\ulcorner \mathbb{R} \urcorner \times \tilde{\mathbb{R}}}$, both ΦK and $\Phi K^{\complement}$.
- Deduce that the diagonal is not a measurable set in $\ulcorner \mathbb{R} \urcorner \times \tilde{\mathbb{R}}$.

(If you find a shorter proof that the diagonal is not measurable, please let me know!) ◿

References