B Lebesgue measurability

Measure theory is based on measurable sets, and the Borel sets of real numbers is the minimal collection of these sets. While the Borel sets are closed under many operations, they are not closed under all of them, and measure theorists and descriptive set theorists investigate other, more general, classes of subsets: analytic sets, universally measurable sets, and the Lebesgue sets. Nonetheless, in this batch of exercises we'll see that the extra level of generality Lebesgue measurability offers, which subsumes the other notions, doesn't get around Aumann's theorem: classical measure theory seems incompatible with function-spaces.

In the process, we'll use measures, measure spaces, and the Lebesgue measurable sets. These concepts come up in the context of higher-order measure theory, and these exercises may serve as classical tutorial to these concepts.

An *outer measure* $\boldsymbol{\lambda}^*$ on a set X is a function $\boldsymbol{\lambda}^* : \wp X \to \mathbb{W}$, i.e., an assignment of a non-negative, potentially infinite, real value to *every* subset, that is moreover *monotonically σ-subadditive*: for every countable set of subsets $I \subseteq_{\aleph_1} \wp X$, and every $A \subseteq \bigcup_{B \in I} B$, we have $\boldsymbol{\lambda}^* A \leq \sum_{B \in I} \boldsymbol{\lambda}^* B$.

◸**B.1.** Let $\boldsymbol{\lambda}^*$ be an outer measure on a set X. Show:

- The empty set has null outer measure: $\boldsymbol{\lambda}^* \varnothing = 0$.
- Monotonicity: $A \subseteq B \implies \boldsymbol{\lambda}^* A \leq \boldsymbol{\lambda}^* B$.
- σ-subadditivity: for every countably infinite family of subset $\vec{A} \in (\wp X)^{\mathbb{N}}$ we have $\boldsymbol{\lambda}^* \left(\bigcup_{i \in \mathbb{N}} A_i\right) \leq \sum_{i \in \mathbb{N}} \boldsymbol{\lambda}^* A_i$.
- Every function $\boldsymbol{\lambda}^* : \wp X \to \mathbb{W}$ satisfying these three conditions is an outer measure. ◿

A *measure* $\boldsymbol{\lambda}$ on a measurable space X is a non-negative, σ-additive function, i.e., for every countable set I and I-indexed family of pairwise-disjoint measurable sets $\langle U_i \in \mathcal{B}_X \rangle_{i \in I}$, we have: $\boldsymbol{\lambda} \left(\bigcup_{i \in I} U_i\right) = \sum_{i \in I} \boldsymbol{\lambda} U_i$. A *measure space* $\Omega = \left\langle \llcorner \underset{\mathbf{Meas}}{\Omega} \lrcorner, \boldsymbol{\lambda}_\Omega \right\rangle$ is a measurable space $\llcorner \underset{\mathbf{Meas}}{\Omega} \lrcorner$ and a measure on it, and similarly an *outer measure* space is a measurable space with an outer measure on it.

Every measure space has an outer measure space on its sets of points. This is the only example of interest. Let Ω be a measure space. Define a function $\boldsymbol{\lambda}^*_\Omega : \wp \llcorner \underset{\mathbf{Set}}{\Omega} \lrcorner \to \mathbb{W}$ by setting, for every $A \in \wp \llcorner \Omega \lrcorner$:

$$\boldsymbol{\lambda}^* A \coloneqq \inf \left\{ \boldsymbol{\lambda} U \,\middle|\, U \in \mathcal{B}_{\llcorner \underset{\mathbf{Meas}}{\Omega} \lrcorner}, U \supseteq A \right\}$$

So $\boldsymbol{\lambda}^* A$ is the least measure we can assign to A by approximating it from the outside with a measurable set. Hence the name — outer measure.

◸**B.2.** Show that $\boldsymbol{\lambda}^*_\Omega$ is an outer measure on $\llcorner \underset{\mathbf{Set}}{\Omega} \lrcorner$, and that it extends $\boldsymbol{\lambda}$: for every $U \in \mathcal{B}_\Omega$, we have $\boldsymbol{\lambda}^* U = \boldsymbol{\lambda} U$. ◿

◸**B.3.** Show that, for every $A \in \wp \llcorner \Omega \lrcorner$ there is some measurable subset $U \in \mathcal{B}_\Omega$, $U \supseteq A$, satisfying $\boldsymbol{\lambda}^* A = \boldsymbol{\lambda} U$. ◿

Let Ω be an outer measure space. A subset $E \subseteq \llcorner \Omega \lrcorner$ is *outer measurable* when, for every $A \subseteq \llcorner \Omega \lrcorner$ we have:

$$\boldsymbol{\lambda}^* A = \boldsymbol{\lambda}^* (A \cap E) + \boldsymbol{\lambda}^* (A \cap E^{\complement})$$

B.4. Let Ω be an outer measure space. For every subset $E \subseteq \llcorner\Omega\lrcorner$, E is outer measurable iff for every $A \subseteq \llcorner\Omega\lrcorner$ we have: $\boldsymbol{\lambda}^* A \geq \boldsymbol{\lambda}^*(A \cap E) + \boldsymbol{\lambda}^*(A \cap E^{\complement})$.

B.5. Let Ω be a measure space. Show that every measurable set $U \in \mathcal{B}_\Omega$ is outer measurable in the associated outer measure space.

B.6. Let Ω be an outer measure space.

- The outer measurable subsets of an outer measure space form a σ-algebra $\mathcal{G}_\Omega$.
- The outer measure $\boldsymbol{\lambda}^*$ restricts to a measure on $\left\langle \llcorner\Omega\lrcorner, \mathcal{G}_\Omega \right\rangle$.

We denote the resulting measure space by $\overline{\overline{\Omega}} := \left\langle \left\langle \llcorner\Omega\lrcorner, \mathcal{G}_\Omega \right\rangle, \boldsymbol{\lambda}^* \right\rangle$.

The *Lebesgue subsets* of $\mathbb{R}$ are the outer measurable subsets w.r.t. the Lebesgue measure. The process: measure space $\Omega \mapsto$ outer measure space $\left\langle \llcorner\Omega\lrcorner, \boldsymbol{\lambda}^* \right\rangle \mapsto$ measure space $\overline{\overline{\Omega}}$ seems like it enhances the space with many more measurable sets. What we'll show next is that these sets aren't too far off from the measurable sets we started with.

A *null* set in a measure space Ω is a subset $Z \subseteq \llcorner\Omega\lrcorner$ that is contained in a 0-measure set: there is some $U \in \mathcal{B}_\Omega$ with $Z \subseteq U$ and $\boldsymbol{\lambda} U = 0$. Let $\mathcal{N}_\Omega$ denote the set of $\boldsymbol{\lambda}$-null sets.

B.7. The null subsets form an ideal: If Z is a null set and $U \subseteq Z$ is any subset, then U is also a null set. Therefore they are closed under non-empty intersections. The null subsets are closed under countable unions.

B.8. Consider the Borel space $\mathbb{R}$ and the Lebesgue measure $\boldsymbol{\lambda}$. Show that there is a $\boldsymbol{\lambda}$-null set that is not Borel measurable.

B.9. Show that every null set is outer measurable.

B.10. Let Ω be a measure space. Prove Ω and $\overline{\overline{\Omega}}$ have the same null sets: $\mathcal{N}_\Omega = \mathcal{N}_{\overline{\overline{\Omega}}}$.

Let Ω be a measure space. A *negligible* measurable subset is a measurable subset $U \in \mathcal{B}_\Omega$ such that, for every measurable subset $V \subseteq U$, we have $\boldsymbol{\lambda} V = 0$ or $\boldsymbol{\lambda} V = \infty$. Non-null negligible measurable subsets are sometimes called 'atomic sets of infinite measure', and Vákár and Ong (2018) call the negligible sets 0-∞-sets. While it may seem strange to call a set of potentially infinite measure negligible, in the context of *integration*, a Lebesgue integrable function must vanish almost everywhere on negligible sets:

B.11. Let U be a negligible measurable subset in a measure space Ω. Let $\varphi : \Omega \to \mathbb{W}$ be a Lebesgue integrable random variable, i.e., a function with a finite expectation $\int \boldsymbol{\lambda} \varphi < \infty$. Show that $\boldsymbol{\lambda} \{\omega \in U | \varphi\omega \neq 0\} = 0$.

A *negligible* subset is a set contained in a negligible measurable subset, and we denote the set of negligible subsets by $\overline{\mathcal{N}}_\Omega$.

B.12. Let Ω be a measure space and consider the scaled measure $\infty \odot \boldsymbol{\lambda}$. Show that:

- Every measurable set U is negligible in the scaled measure, and therefore every subset is negligible.
- A subset is null in the scaled measure iff it is null in Ω.

B.13. Consider the Lebesgue measure on $\mathbb{R}$. Show every negligible subset is null.

B.14. The negligible subsets generalise the null sets and have analogous properties:

- The negligible subsets form an ideal.
- The negligible subsets are closed under countable unions.
- Every negligible subset is outer measurable.
- Every negligible subset of finite outer measure is null. ◿

The *completion* of a measure space Ω is the following measurable space $\overline{\Omega}$:

- It has the same points $\llcorner\overline{\langle X, \boldsymbol{\lambda}\rangle}\lrcorner := \llcorner X \lrcorner$.
- Its σ-algebra is generated by the measurable sets and the null sets: $\mathcal{B}_{\overline{\Omega}} := \sigma(\mathcal{B}_X \cup \mathcal{N})$.

◸**B.15.** Show that the following are equivalent for a subset $U \subseteq \llcorner\Omega\lrcorner$:

- U is measurable in the completion $\overline{\Omega}$
- There is a measurable set $V \in \mathcal{B}_\Omega$ and a null set $Z \in \mathcal{N}_\Omega$ such that $U = V \cup Z$.
- There is a measurable $V \in \mathcal{B}_X$ such that $U \setminus V$ is null. ◿

◸**B.16.** Let E be an outer measurable subset in a measure space Ω. Show that if E has finite outer measure, then:

- There are measurable $U, V \in \mathcal{B}_\Omega$ with $U \subseteq E \subseteq V$ and $\boldsymbol{\lambda} U = \boldsymbol{\lambda}^* E = \boldsymbol{\lambda} V$.
- $E = E_{\mathcal{B}} \cup E_{\mathcal{N}}$ where $E_{\mathcal{B}} \in \mathcal{B}_\Omega$ and $E_{\mathcal{N}} \in \mathcal{N}$. ◿

A measure space Ω is *σ-finite* when there is a countable measurable partition $\llcorner\Omega\lrcorner = \biguplus_{i \in I} \Omega_i$ for which every subset has $\boldsymbol{\lambda}\Omega_i = 0$.

◸**B.17.** Show that in a σ-finite space Ω, the outer measurable sets coincide with the completion σ-algebra: $\mathcal{B}_{\overline{\overline{\Omega}}} = \mathcal{B}_{\overline{\Omega}}$ ◿

Let $\mathbb{R}_{\boldsymbol{\lambda}}$ be the measurable space over the reals with the Lebesgue σ-algebra. By the last few exercises, every Lebesgue measurable set on the reals is a Borel set apart from a null set of points. Similarly, a Lebesgue measurable function $f : \underset{\mathbf{Meas}}{\llcorner \mathbb{R}_{\boldsymbol{\lambda}} \lrcorner} \to \mathbb{R}$ is almost-everywhere equal to a Borel measurable function $g : \mathbb{R} \to \mathbb{R}$:

◸**B.18.** Let X be a measurable space whose σ-algebra is *countably generated*, i.e., there is a countable set $\mathcal{U} \subseteq \mathcal{B}_X$ such that $\mathcal{B}_X = \sigma(\mathcal{U})$. For every Lebesgue measurable function $f : \mathbb{R}_{\boldsymbol{\lambda}} \to X$ there is a Borel measurable function $g : \mathbb{R} \to X$ such that $f(x) = g(x)$ $\boldsymbol{\lambda}(\mathrm{d}x)$-almost certainly. ◿

So the class of Lebesgue measurable functions is not profoundly different from the class of Borel measurable functions, especially as far as integration is concerned.

We are now ready to prove the Lebesgue-measurable version of Aumann's theorem:

▸ **Theorem** (Aumann's theorem for Lebesgue measurable evaluation)**.** *There is no σ-algebra on $\mathcal{B}_{\mathbb{R}}$ making the membership relation $[- \in -] : \mathcal{B}_{\mathbb{R}} \times \mathbb{R}_{\boldsymbol{\lambda}} \to \mathbb{2}$ measurable. Similarly, there is no σ-algebra on $\mathbf{Meas}(\mathbb{R}, \mathbb{R})$ making evaluation $\mathrm{eval} : \mathbf{Meas}(\mathbb{R}, \mathbb{R}) \times \mathbb{R}_{\boldsymbol{\lambda}} \to \mathbb{R}$ measurable.*

It suffices prove that the discrete σ-algebra on $\mathcal{B}_{\mathbb{R}}$ doesn't make the membership predicate measurable:

◸**B.19.** Assume that $[- \in -] : \mathcal{B}_{\mathbb{R}} \times \mathbb{R}_{\boldsymbol{\lambda}} \to \mathbb{2}$ is not measurable when we equip $\mathcal{B}_{\mathbb{R}}$ with the discrete σ-algebra. Show the following.

- The membership predicate is not measurable w.r.t. every σ-algebra on $\mathcal{B}_{\mathbb{R}}$.

- Evaluation $\mathrm{eval} : \mathbf{Meas}(\mathbb{R},\mathbb{R}) \times \mathbb{R}_{\boldsymbol{\lambda}} \to \mathbb{R}$ is not measurable w.r.t. every σ-algebra on $\mathbf{Meas}(\mathbb{R},\mathbb{R})$. ◿

From this point, we assume to the contrary that $[- \in -] : \mathcal{B}_{\mathbb{R}} \times \mathbb{R}_{\boldsymbol{\lambda}} \to 2$ is measurable. Let:

$$\mathcal{U}_0 \coloneqq [- \in -]^{-1}[\mathbf{true}] = \{\langle U, x\rangle \in \mathcal{B}_{\mathbb{R}} \times \mathbb{R} | x \in U\} \in \mathcal{B}_{\mathcal{B}_{\mathbb{R}} \times \mathbb{R}_{\boldsymbol{\lambda}}} = \wp\mathcal{B}_{\mathbb{R}} \otimes (\mathcal{B}_{\mathbb{R}} \cup \mathcal{N}_{\mathbb{R}})$$

Let $e_{\mathcal{B}} : \boldsymbol{b} \twoheadrightarrow \mathcal{B}_{\mathbb{R}}, e_{\mathcal{N}} : \boldsymbol{n} \twoheadrightarrow \mathcal{N}_{\mathbb{R}}$, and $e_{\wp} : \boldsymbol{p} \twoheadrightarrow \wp\mathcal{B}_{\mathbb{R}}$ be enumerations of the Borel sets, null sets, and powerset-over-Borel-sets of reals, respectively. Then we also have an enumeration of a generating family for the box σ-algebra of the product space $\mathcal{B}_{\mathbb{R}} \times \mathbb{R}_{\boldsymbol{\lambda}}$:

$$e : \boldsymbol{p} \times (\boldsymbol{b} \uplus \boldsymbol{n}) \twoheadrightarrow [\wp\mathcal{B}_{\mathbb{R}}] \times [\mathcal{B}_{\mathbb{R}} \cup \mathcal{N}_{\mathbb{R}}] \coloneqq \{\mathcal{U} \times E | \mathcal{U} \subseteq \mathcal{B}_{\mathbb{R}}, E \in \mathcal{B}_{\mathbb{R}} \cup \mathcal{N}_{\mathbb{R}}\}$$
$$e \coloneqq (e_{\wp}(\pi_1 -)) \times ([e_{\mathcal{B}}, e_{\mathcal{N}}]\,(\pi_2 -))$$

By Ex.A.4 the σ-term interpretation function $[\![-]\!]\,e$ is surjective, and so there is some σ-term t such that $\mathcal{U}_0 = [\![t]\!]\,e$. Let $V_0 \coloneqq \mathrm{supp}\,t$, and then $V_0 \subseteq \boldsymbol{p} \times (\boldsymbol{b} \uplus \boldsymbol{n})$ is a countable enumeration of the variable names that appear in t, and we may restrict e to $e_0 : V_0 \to [\wp\mathcal{B}_{\mathbb{R}}] \times [\mathcal{B}_{\mathbb{R}} \cup \mathcal{N}_{\mathbb{R}}]$, and consider t as a term t_0 over V_0 such that $[\![t_0]\!]\,e_0 = \mathcal{U}_0$.

Let:

- $N \coloneqq \bigcup_{(p,\iota_2 n) \in V_0} e_{\mathcal{N}} \in \mathcal{N}$;
- Z a 0-measure Borel set with $N \subseteq Z$; and
- $S \coloneqq \mathbb{R} \setminus Z$ equipped with the Borel-subspace σ-algebra.

◸**B.20.** Show that N is indeed a null set, so that Z exists. Show that S is an uncountable Borel set. ◿

Define $e_1 : V_0 \to \wp\mathcal{B}_{\mathbb{R}} \otimes \mathcal{B}_S$ by setting:

$$e_1(p, \iota_1 b) \coloneqq (e_{\wp} p) \times (e_{\mathcal{B}} b \cap S) \qquad e_1(p, \iota_2 n) \coloneqq \varnothing$$

By re-interpreting the σ-term t_0 with e_1, we have a measurable set $[\![t_0]\!]\,e_1 \in \mathcal{B}_{\mathcal{B}_{\mathbb{R}} \times S}$.

◸**B.21.** Show that for every $U \in \mathcal{B}_{\mathbb{R}}$ and $s \in S$, we have $\langle U, s\rangle \in [\![t_0]\!]\,e_0$ iff $\langle U, s\rangle \in [\![t_0]\!]\,e_1$. ◿

The last ingredient is to note that, by the original Aumann's theorem, there is no σ-algebra on $\mathcal{B}_S$ that makes the membership predicate $[- \in -] : \mathcal{B}_S \times \mathbb{R} \to 2$ measurable.

◸**B.22.** Use this last fact to get the desired contradiction. ◿

References

Matthijs Vákár and Luke Ong. On s-finite measures and kernels. *arXiv preprint arXiv:1810.01837*, 2018.